Linear Programming (LP)—Simplex Method (Handout)

1. Simplex Algorithm(Maximization Problem)

An *algorithm* is simply a process where a systematic procedure is repeated (iterated) over and over again until the desired result is obtained.

The Simplex algorithm exploits the properties of the LP solutions by examining only relatively few of the promising extreme feasible solutions and stopping as soon as one of them passes this optimality test.

Step 1: *Convert the LP to standard form*

- All variables (x, *s*, *e*, *a*) must be nonnegative.
- All right-hand sides (*b*) must be non-negative (≥ 0).
- All (FUNCTIONAL) constraints must be equality constraints.

Step 2: *Initialization*
Select the *initial* nonbasic variables and the basic variables and obtain an initial **basic feasible solution** (bfs) if possible from the standard form.
Geometrically, this is equivalent to starting at any convenient corner-point that satisfies all constraints (including the nonnegativity ones).

Step 3: Optimality test [Rule 1]
The current bfs is optimal if and only if every coefficient in **row 0** (also called reduced cost) is nonnegative (≥ 0). [Stop if that's the case and note the optimal bfs]

Step 4: Iteration(s)

a. Determine the **entering** basic variable by selecting the nonbasic variable with the negative coefficient having the largest absolute value. (The entering basic variable should be the one that has the largest positive rate of change on Z, since we are improving the objective function over the current one at a faster rate.) Put a box around the column below this coefficient, and call this the ***pivot column***. [Rule 2]

b. Apply the ***minimum ratio test*** (MRT) to identify the leaving variable (the winner of the minimum ratio test). Select the basic variable that reaches zero first as the entering variable is increased. [Rule 3]

   Formally: For each Row *i*, ***i* ≥ 1**, where there is a ***strictly positive*** "entering variable coefficient" (coefficients in the pivot column that are not zero or negative), compute the ratio of the Right Hand Side to that of the entering variable coefficient in the pivot column. Choose the ***pivot row*** as being the one with the MINIMUM ratio test value.

c. Use elementary row operations (ERO) (Gauss-Jordan elimination) to obtain the new entering basic variable. In other words, transform the system into a form to get the new bfs.

d. Go to step 3.

2. Simplex Algorithm(Minimization Problem)

Method 1: If the objective function is a minimization, then the same rules will be used but with changing the objective function to maximization. The reason why the two formulations are equivalent is that the smaller Z is, the larger (-Z) is, so the solution that gives the smallest value for the objective function in the entire feasible region must also give the largest value of (-Z) in this region.
Method 2:
1- **Modify the rule of optimality** as follows: If all nonbasic variables in row 0 have nonpositive coefficients, then the current BF solution is optimal.
2- **Modify the rule for choosing the entering nonbasic variable** as follows: If any nonbasic variable in row 0 has a positive coefficient, choose the variable with the "most positive" coefficient in row 0 to enter the basis.

3. The Big M Simplex Algorithm

An important condition to using the Simplex algorithm requires writing the LP model in a standard form in which all right-hand-sides are nonnegative. The class of LP models in which all functional constraints are of the ≤ type ensures the availability of starting (initial) basic variables that satisfy all non-negativity constraints which gives initial bfs.

Reminder: (**A basic variable for an equation is a variable whose coefficient in the equation is +1 and whose coefficient in all other equations of the problem is 0**.)

In general, an initial (starting) bfs may not be readily available, hence we must modify the original LP model to ensure the availability of initial basic feasible variables, one for each functional constraint.

Description of Big M Method

Step 1: *Convert the LP to standard form*

- All variables (x) must be nonnegative. ***This requires that each constraint with a negative RHS by multiplied by (-1). Be careful to reverse the direction of inequalities when you do this***.
- All right-hand sides (b) must be non-negative (≥ 0).
- All (FUNCTIONAL) constraints must be equality. **This means adding slack and excess variables to ≤ and ≥ constraints, respectively. No slack or excess variables is required for = constraints**.

Step 2: *Initialization and modification*

Select the initial nonbasic variables and the basic variables and obtain an initial basic feasible solution if possible from the standard form.

An *artificial* variable a_i must be added to each constraint *i* that does not have a basic variable. Also add the sign restriction $a_i \geq 0$.

Step 3: Let M denote a very large positive number. If the LP is a min problem, add (for each artificial variable) Ma_i to the objective function. If the LP is a max problem, add (for each artificial variable) $-Ma_i$ to the objective function.

Step 4: Because each artificial variable will be in the starting basis, all artificial variables must be eliminated from row 0 before beginning the simplex. To guarantee that the optimal solution of the modified LP model is the optimal solution of the original LP model, the optimal solution of the modified model should not contain any artificial variable as BF solution. Once an artificial variable leaves the basis we no longer need it, however we often maintain the artificial variables in all tableaus for sensitivity analysis purposes.

How to spot an infeasible LP: If any artificial variables are positive in the optimal solution, the original problem is infeasible.

4. Special Case 1: Alternative optimal solution

Observation: At optimality, all nonbasic variables have nonnegative (≥0) coefficients in row 0, hence the bfs is optimal.

- If there is no nonbasic variable with a zero coefficient in row 0 of the optimal tableau, then the LP has a unique optimal solution.
- Even if there is a nonbasic variable with a zero coefficient in row 0 of the optimal tableau, it is possible that the LP may not have an alternative optimal solutions.

5. Special Case 2: Unbounded LPs

An unbounded LP for a maximization problem occurs when a nonbasic variable with a negative coefficient in row 0 has a nonpositive coefficient in each constraint. The objective function value improves with each iteration

6. Special Case 3: Degenerate LP model

When the solution has a basic variable (namely s1) which is equal to zero, we say that the basic solution is **degenerate**.

Degeneracy reveals from practical standpoint that the model has at least one redundant constraint.