Linear Programming (LP)—Introduction and Formulation

Linear program model uses a mathematical (or an optimization) model to describe the problem of concern. It contains a single objective function and a number of constraints. All functions are linear.

1. Components of an Linear Program (LP)

Decision Variables: The decision variables represent (unknown) activity levels **or** choices/decisions to be made. This is in contrast to problem parameters (uncontrollable factors), which are values that are either given or can be simply calculated from what is given.

Objective: *Every linear program has an objective*. The objective function quantifies the decision consequences to be maximized (MAX) or minimized (MIN) making some decisions preferred to others. This objective has to be linear in the decision variables, which means it must be the sum of constants times decision variables.

Constraints: *Every linear program has constraints that limit* the range of values *of the decision variable*s. These constraints represent restrictions and limitations on the decisions that to can be made. They can be physical, economic, policy, or environmental etc.

- Inequality: Less than or equal (≤) constraint (resource), Greater than or equal (≥) constraint (requirement).
- Equality: Equal (=). Equality can be converted to inequality.

Feasible solutions (decisions): Any decisions that satisfy all the constraints.

Optimal solution (decision): The one among all feasible decisions that has the optimum objective function.

2. LP Problem Formulation

Problem formulation or modeling is the process of translating a verbal statement of a problem into a mathematical statement:

a. Identify the decision variables, and if possible assign a symbol to each one.

b. Describe the objective function.

c. Know what limits the decisions (constraints), and express each constraint in terms of decision variables.

   i. Main or functional constraints.

   ii. Variable-type constraints or sign restrictions (nonnegative, nonpositive, or unrestricted—or free).

d. Calculate the objective function coefficients of the decision variables.

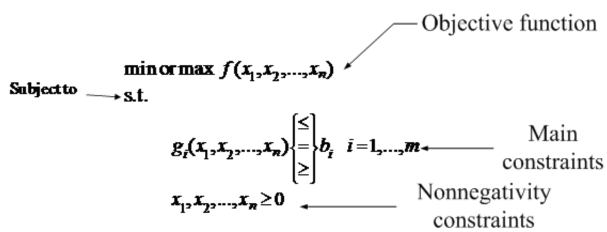
e. Calculate the left-hand-side (LHS) coefficients of each constraint. (Technological coefficients)

f. Express the objective function in terms of variables. (must be a linear function)

g. Express the constraints in terms of variables. (must be a linear function)

3. Mathematical Form of a Linear Program

LP general format



- Where $f, g_1, \ldots, g_m$ are given functions of decision variables.

Objective function and constraints functions must be linear, i.e. variables are only multiplied or divided by scalars and added or subtracted from each other. Neither variables multiplication nor division is allowed. Each variable appears in a separate term together with its coefficient.

LP general format (Maximizing LP)

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_l + \ldots + a_{mn}x_n \le b_m$$

$$\text{and} \qquad x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$$

LP general format (Minimizing LP)

Resources levels

n variables
m constraint

$$\text{Minimize } W = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

Subject to

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ge d_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \ge d_2$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_l + \ldots + a_{mn}x_n \ge d_m$$

$$\text{and} \qquad x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0$$

Objective function coefficients of decision variables

LHS coefficients

Demands levels

Properties of LP:

Proportionality: the contribution of each decision variable in both the objective function and constraint is directly proportional to the value of the variable. (Nonlinear functions are not allowed)

Additivity: the total contribution of all the variables in the objective function and in the constraint is direct sum of the individual contributions of each variable.
If two decision variables compete, that is an increase in one would necessary require a decrease in the other, that would violate this property.

Certainity: all parameters (technological) and objective function coefficients are known with certaininty.

LP Solution Methods

4. Definitions and terminologies

D1: **Feasible (set) region**: a collection of values for the decision variables that satisfy all LP's model constraints.

D2: Optimal solution is a feasible choice for decision variables with objective function value at least equal to that of any other solution satisfying all constraints:

- For a maximization problem, an optimal solution to an LP is a point in the feasible region with the largest function value.
- For a minimization problem, an optimal solution to an LP is a point in the feasible region with the smallest objective function value.

D3: **Optimal value** in an optimization model is the objective function value of any optimal solutions**.**

D4: Binding constraint: a constraint is binding if the left-hand-side and the right-hand-side of the constraint are equal when the optimal values of the decision variables are substituted into the constraint. The binding constrain is said to be active constraint.

D5: **Boundary point**: a feasible solution to an LP is a boundary point if at least one inequality constraint that can be strict for some feasible solutions is satisfied as equality at the given point (i.e., the constraint is said to be *active*). A feasible solution is an interior point if no such inequalities are active.

Convex set: a feasible set of an optimization problem is convex if the line segment between every pair of this set falls entirely within the feasible region.

Extreme point: a point *P* in a convex set *S* is an extreme point if each line segment that lies in *S* and contains the point *P* has *P* as an endpoint of the line segment. If the convex set is a **polygon**, an extreme point is called a corner point, and it will be the vertices of the polygon. A feasible extreme point is a basic feasible solution to the LP model.

Multiple optimal solutions occur when there are an infinite number of optimal solutions--i.e., infinite feasible optimal extreme points are optimal.

5. Graphical Solution Procedure

1- Plot every constraint in the LP model by setting it to "equality" form, and then identify the set of points that satisfies that constraints.

2- Determine the *set of all points that satisfies all the LP's constraints and the variable-type constraints (i.e., sign restrictions)—**feasible region***.

3- Search for the point in the feasible region with the largest (or smallest) objective function value in a maximization problem (or minimization)—i.e., search for optimal solution.

- Identify the ***convex set***: A feasible set of an optimization problem is convex if the line segment between every pair of this set falls entirely within the feasible region.
- Plot the objective function equation (isoprofit in maximization or isocost in minimization LP model) by selecting any arbitrary point in the feasible region and calculating its z-value. Since the slope of the isoprofit/isocost-line is constant we can draw identical parallel lines.
- Move the objective function line toward ***Optimality***. Move it in the direction which would improve its value (increase the value of the objective function if maximizing or decrease the value of the objective function if minimizing).

6. LP Model Solution Special Cases

The feasible region for a linear programming problem can be nonexistent, a single point, a line, a polygon, polyhedron, or an unbounded area.

In the graphical method, if the objective function line is parallel to a boundary constraint in the direction of optimization, there are **alternative optimal solutions**, with all points on this line segment being optimal.

A linear program which is over-constrained so that no point satisfies all the constraints is said to be **infeasible**.

7. Standard Form of an LP

The requirements for an LP to be in a standard form are:

a. have only equality main constraints;

b. <u>have only nonnegative variables</u>; and

c. all right-hand-sides (b_i) must be nonnegative (≥ 0).

d. have objective function and main constraints simplified so that variables appear at most once. For constraints, variables should be on the left-hand side, and any constant term (possibly zero) appears on the right-hand side.

To convert an LP into a Standard Form utilize **slack/surplus** variables. I will discuss this in class.

Slack and **surplus** variables represent the **difference** between the left and right sides of the constraints.

8. Geometric Properties of an LP [Very Important]

The feasible region can be a point, a line, an area, or nonexistent.

Basic variable: is a variable that appears with a coefficient of 1 in a single equation and a coefficient of 0 in all other equations.

Definition 1: In a problem with n variables and m constraints (assuming $n \geq m$) any basic solution is obtained by setting $n - m$ variables equal to 0 and solving for the values of the remaining m variables, assuming there is a unique solution for the remaining m variables (their columns are linearly independent).

9. Properties of an LP Model Solution

Theorem 1: The feasible region for any LP will be convex set.

Theorem 2: A point in the feasible region of an LP is an extreme point (called extreme point feasible) if and only if it is a ***basic feasible solution*** (bfs) to the LP.

Definition 1: Any basic solution to ***Eq. 1*** in which all variables *are nonnegative* is a **basic feasible solution** (or, bfs).

Theorem 3: If an LP has a unique optimal solution then it has an optimal bfs. This implies that the optimal solution is a feasible extreme point (or a **CPF solution**).

Theorem 4: If an LP has a multiple optimal solutions then at least two must be adjacent CPF solutions (or feasible extreme points).

Property 1: There are only a finite number of CPF solutions.

Property 2: If a CPF solution has no adjacent CPF solutions that are better, then there are no better CPF solutions anywhere.