**UNIVERSITY OF WATERLOO**
**MANAGEMENT ENGINEERING PROGRAM**
**DEPARTMENT OF MANAGEMENT SCIENCES**

MSCI 331 *Introduction to Optimization*

**Linear Programming (LP)—Summary of Important Sensitivity Analysis Concepts**

## 1. The Importance of Sensitivity Analysis

In sensitivity analysis we determine how values of input parameters and activities change and their effect on changing the optimal solution.

- Tells us which parameter is most crucial.
- Enables us to update optimal solution when LP's parameters or activities change without repeating all calculations (in big models that is a big issue).
- One of the assumptions of LP formulation is certainty. Hence sensitivity analysis enables us to test the robustness of the model. (Which parameters are crucial and which are sensitive to small variations).
- Provides us (analysts and managers) with more understanding (insights) about the problem we are modeling.
- Allows us to answer some "what-if" questions:
  1. What if cost or demand changed, can we sill trust mathematically optimal answers and take decisions according to obtained results?
  2. What if our customers changed their consumption behavior, when we need to respond and how can we respond?

*Graphical method of sensitivity analysis when we have only two variables to examine*:

- Objective function coefficient sensitivity: determine *range of optimality for each decision variable coefficient.*
- Right-hand-side sensitivity (shadow prices): determine *range of feasibility for each of the RHS of constraints.*
- Introducing more constraints (affects the geometry of feasible region) or introducing more activities (affects the competition over resources)

*Computer sensitivity for any number of variables to examine*:

- Objective function coefficient sensitivity
- Changes is demand or resource levels (shadow prices)

Typical questions:

- Consider how changes in the objective function coefficient may affect the optimal solution.
- Find the range of optimality of the objective function coefficients.
- What is the range of profit contribution of a specific decision or activity for which the current optimal solution remains the same.
- If cost changes and/or marginal profit contribution of an activity changes what will be the new level of the activity.
- Should we acquire more of a specific resource and how much we should pay for it.

## 2. Range of Optimality for each Objective Function's Coefficient

Range of optimality for a **basic variable**: the range of values of the basic variable coefficient in the objective function for which the current set of basic variables remains optimal.

Range of optimality for a **nonbasic variable**: the range of values of the nonbasic variable coefficient in the objective function that keeps the variable nonbasic.

**Range of optimality** for each objective function coefficient provides us with a measure of how much we can improve the profit contribution (in case of maximization) or cost (in case of minimization) of a decision variable without losing the optimality we already have.

Managers should focus on those objective coefficients that have a narrow range of optimality and coefficients near the endpoints of the range.

In general, for the case of two decision variables, $z = c_1x_1 + c_2x_2 = \text{constant}$ , then $x_2 = \frac{\text{constant}}{c_2} - \frac{c_1}{c_2}x_1$

The slope of an objective function line is $-c_1/c_2$, and the slope of a constraint, $a_1x_1 + a_2x_2 = b$, is $-a_1/a_2$. Hence, find the range of values for $c_1$ (with $c_2$ staying constant) such that the objective function line slope lies between binding constraints while maintaining the current optimal solution (i.e., same basic variables and same values for these basic variables). Notice that changing in any of the objective function coefficient does not alter or affect the characteristic (geometry) of the feasible region.

When finding the range of optimality for a coefficient you will find upper and lower limits and from that find the allowable increase and decrease. The upper and lower limits also give you the range of optimality.

Also notice that while within the range of optimality, the optimal basic feasible solution (the basic variables and their values) remain the same, the objective function value will change according to the change in the coefficient.

## 3. Range of Feasibility for RHS Value of Each Constraints

**Range of feasibility**: the range of values over which a RHS may vary without affecting the optimal set of basic variables. The values of the basic variables in the solution will change, but the same set of variables will remain basic.

Graphically (in the case of two decision variables), the range of feasibility is determined by finding the values of a right hand side coefficient such that the same lines that determined the original optimal solution continue to determine the optimal solution for the problem:

- Within the range of feasibility, the current basic solution remains optimal, but the values of the decision variables and the objective function values will change.

- As RHS of a constraint changes, other constraints will become binding and limit the change in the value of the objective function.

## 4. Dual Price for Each Constraints

First, a note about binding and nonbinding constraints. A binding constraint is one in which for specific values of the variables the right-hand-side (RHS) and left-hand-side are equal. The following cases illustrate:

- Case 1: "≤" or "≥" constraints
  Suppose we have the following constraint

  $$3x_1 + 5x_2 \leq 60$$

  For $x_1 = 10, \quad x_2 = 5$, we know that:

  1. Not all the 60 units are being used. Hence we have a slack of 5 units.
  2. The RHS and LHS are not equal. The constraint is not binding.
  3. There is no sense of acquiring additional quantities of the resource.

  For $x_1 = 10, \quad x_2 = 6$:

  1. The RHS and LHS are equal.
  2. The constraint is binding, or active.

  Hence the instance the feasible solution lies on a constraint, the constraint becomes binding.

- Case 2: "=" constraint
  This is a hard constraint, and it's always binding in an LP model.

**Dual Price**: the dual price for the *j*th constraint of an LP is the amount by which the optimal value is **improved** (increased in Max and decreased in Min) if the right-hand-side of the *j*th constraint is increased by 1unit. (Called also Shadow Price in the book).

A dual price of a constraint is determined by adding +1 to the right hand side value of the constraint in question and then re-solve for the optimal solution in terms of the *same* binding constraints.

- The shadow price is equal to the difference in the values of the objective function between the new and original problems.

  **For Maximization LP**: New optimal z-value – Old optimal z-value=constraints' *jth* shadow price x $\Delta b_j$

  **For a Minimization LP**: New optimal z-value – Old optimal z-value= – constraints' *j*th shadow price x $\Delta b_j$

- The shadow price for a nonbinding constraint is 0.
- A negative shadow price indicates that the objective function will become worse if the RHS is increased.

[*Refer to class notes for the observations or axioms that I wrote on the whiteboard*]

## 5. Reduced Cost

Gives information about the amount by which an objective function coefficient for a nonbasic variable has to change to affect the the LP's optimal solution.

For any nonbasic variable $x_i$, the reduced cost is the amount by which the objective function coefficient of $x_i$ must be **improved** before the LP will have an optimal solution in which $x_i$ is a basic variable.

a. If the objective function coefficient of $x_i$ is improved by its reduced cost, then the LP will have alternative optimal solutions.

   **At least one in which $x_i$ is a basic variable, and at least one in which $x_i$ is not a basic variable.**

b. If the objective function coefficient of a nonbasic variable is improved by more than its reduced cost, then (barring degeneracy) any optimal solution to the LP will have $x_i$ as a basic variable and $x_i$ >0.

## 6. Generic Sensitivity Analysis

**Changes in objective function coefficients:**

| Model Form (Primal) | Coefficient Increase | Coefficient Decrease |
|---|---|---|
| Maximize objective | Same or better | Same or worse |
| Minimize objective | Same or worse | Same or better |

**Changes to the RHS of constraints**:

The range of feasibility for a right hand side coefficient is the range of that coefficient for which the shadow price remains unchanged.

Changes in LP model RHS coefficients affect the feasible space as the following two principles:

1. *Relaxing* a constraint admits new feasible solutions.
    - Relaxing the constraints of an optimization model either leaves the optimal value unchanged or makes it better.
2. *Tightening* the constraints restricts the feasible solutions.
    - Tightening the constraints either leaves the optimal value unchanged or makes it worse.

| Constraint Type | RHS Increase | RHS Decrease |
|---|---|---|
| Supply $\leq$ | Relax | Tighten |
| Demand $\geq$ | Tighten | Relax |

**Changes to the LHS of constraints**:

1. A larger coefficient on the left-hand side of a $\geq$ constraint makes it easier (for nonnegative variables) to satisfy, and a smaller coefficient make it harder to satisfy.
2. A larger coefficient on the RHS of $\leq$ constraint makes it hard to satisfy.

| Constraint Type | Coefficient Increase | Coefficient Decrease |
|---|---|---|
| Supply $\leq$ | Tighten | Relax |
| Demand $\geq$ | Relax | Tighten |

**Introducing a new constraint**:

- Check directly whether the optimal solution satisfies the constraint.
- If it does, then it would still be the best feasible solution.
- If the current optimal solutions violate the introduced constraint (reduces the feasible region), then introduce the constrain into the final simplex tableau as an additional row.
- Carry on and re-optimize.

**Introducing a new activity**: Leave for students to find out.