

PHASE 1 SIMPLEX METHOD

Consider the following problem with $m = 3$ constraints in $n = 3$ unknowns:

$$\begin{array}{rcll} \text{Maximize} & x_1 & - & x_2 & + & x_3 & & & \\ \text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & \leq & 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & \leq & -5 \\ & -x_1 & + & x_2 & - & 2x_3 & \leq & -1 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

In **standard form** this becomes:

$$\begin{array}{rcll} \text{Maximize} & x_1 & - & x_2 & + & x_3 & & & \\ \text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & & + & x_5 & = & -5 \\ & -x_1 & + & x_2 & - & 2x_3 & & & + & x_6 & = & -1 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & \geq & 0 \end{array}$$

On the previous handout (The Simplex Method Using Dictionaries) an initial BFS was obtained by making the original variables nonbasic (i.e. fixing their value to zero) and the slack variables basic. Using that same approach in this example would yield a basic solution *that would be infeasible* (since $x_5 = -5$, $x_6 = -1$ violate their nonnegativity restrictions!). In such cases we create an artificial (Phase 1) problem, and its initial feasible dictionary, using the following **mutually exclusive and complimentary steps**:

1. start with the **standard form** LP (as above)
2. to each main constraint $i \in \{1, \dots, m\}$ **without** a slack variable, add (subtract) a unique nonnegative **artificial variable** to (from) the LHS if $b_i \geq 0$ ($b_i < 0$) and make this artificial variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.
3. to each main constraint $i \in \{1, \dots, m\}$ **with** a slack variable that would be **nonnegative if all original decision variables are set to zero**, make this slack variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.
4. to each main constraint $i \in \{1, \dots, m\}$ **with** a slack variable that would be **negative if all original decision variables are set to zero**, add (subtract) a unique nonnegative **artificial variable** to (from) the LHS if $b_i \geq 0$ ($b_i < 0$) and make this artificial variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.
5. construct the equations for the Phase 1 initial feasible dictionary by isolating the basic variables (identified above) on the LHS of each main equation. The artificial objective, w , to be **maximized** is the **negative sum** of the artificial variables that were introduced in steps 2 and 4 above. Include the artificial objective as the last equation in the initial feasible dictionary by isolating w on the LHS and by expressing w **in terms of the nonbasic variables (and a constant) only**. **Note; by construction, $w \leq 0$ for all feasible solutions.**

For this example we would construct the following artificial Phase 1 problem

$$\begin{array}{rcll} \text{Maximize} & -x_7 & - & x_8 & & & & & \\ \text{subject to} & 2x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 4 \\ & 2x_1 & - & 3x_2 & + & x_3 & & + & x_5 & - & x_7 & = & -5 \\ & -x_1 & + & x_2 & - & 2x_3 & & & + & x_6 & - & x_8 & = & -1 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & , & x_5 & , & x_6 & , & x_7 & , & x_8 & \geq & 0 \end{array}$$

and the corresponding initial dictionary

$$\begin{array}{rcll} x_4 & = & 4 & - & 2x_1 & + & x_2 & - & 2x_3 \\ x_7 & = & 5 & + & 2x_1 & - & 3x_2 & + & x_3 & + & x_5 \\ x_8 & = & 1 & - & x_1 & + & x_2 & - & 2x_3 & & + & x_6 \\ w & = & -6 & - & x_1 & + & 2x_2 & + & x_3 & - & x_5 & - & x_6. \end{array}$$

Basic feasible solution	$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 4, x_5 = 0, x_6 = 0, x_7 = 5, x_8 = 1, w = -6$
Entering variable	x_2
Leaving variable	x_7
Pivot equation	$x_2 = \frac{5}{3} + \frac{2}{3}x_1 + \frac{1}{3}x_3 + \frac{1}{3}x_5 - \frac{1}{3}x_7$

$$\begin{array}{rcllclll} x_2 & = & \frac{5}{3} & + & \frac{2}{3}x_1 & + & \frac{1}{3}x_3 & + & \frac{1}{3}x_5 & - & \frac{1}{3}x_7 \\ x_4 & = & \frac{17}{3} & - & \frac{4}{3}x_1 & - & \frac{5}{3}x_3 & + & \frac{1}{3}x_5 & - & \frac{1}{3}x_7 \\ x_8 & = & \frac{8}{3} & - & \frac{1}{3}x_1 & - & \frac{5}{3}x_3 & + & \frac{1}{3}x_5 & + & x_6 - \frac{1}{3}x_7 \\[10pt] w & = & -\frac{8}{3} & + & \frac{1}{3}x_1 & + & \frac{5}{3}x_3 & - & \frac{1}{3}x_5 & - & x_6 - \frac{2}{3}x_7 \end{array}$$

Basic feasible solution	$x_1 = 0, x_2 = \frac{5}{3}, x_3 = 0, x_4 = \frac{17}{3}, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = \frac{8}{3}, w = -\frac{8}{3}$
Entering variable	x_3
Leaving variable	x_8
Pivot equation	$x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 - \frac{1}{5}x_7 - \frac{3}{5}x_8$

$$\begin{array}{rcccccccc} x_2 & = & \frac{11}{5} & + & \frac{3}{5}x_1 & + & \frac{2}{5}x_5 & + & \frac{1}{5}x_6 & - & \frac{2}{5}x_7 & - & \frac{1}{5}x_8 \\ x_3 & = & \frac{8}{5} & - & \frac{1}{5}x_1 & + & \frac{1}{5}x_5 & + & \frac{3}{5}x_6 & - & \frac{1}{5}x_7 & - & \frac{3}{5}x_8 \\ x_4 & = & 3 & - & x_1 & & & - & x_6 & & & + & x_8 \\ w & = & 0 & & & & & & & - & x_7 & - & x_8 \end{array}$$

Since $w^* = 0$ this optimal dictionary for the artificial Phase 1 problem yields a feasible vertex of the original problem, namely $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0)$. Furthermore, this Phase 1 dictionary can *easily* be converted into a corresponding *feasible* dictionary for the original problem by:

1. eliminating the artificial variables from all the optimal dictionary equations, and
2. replacing the equation for w with the *original* objective z expressed in terms of the nonbasic variables identified in the optimal artificial Phase 1 dictionary (ie. x_1, x_5 and x_6).

$$z = x_1 - x_2 + x_3 = x_1 - \left(\frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6\right) + \left(\frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6\right) = -\frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6$$
$$\begin{array}{rclclcl} x_2 & = & \frac{11}{5} & + & \frac{3}{5}x_1 & + & \frac{2}{5}x_5 & + & \frac{1}{5}x_6 \\ x_3 & = & \frac{8}{5} & - & \frac{1}{5}x_1 & + & \frac{1}{5}x_5 & + & \frac{3}{5}x_6 \\ x_4 & = & 3 & - & x_1 & & & - & x_6 \\ \\ z & = & -\frac{3}{5} & + & \frac{1}{5}x_1 & - & \frac{1}{5}x_5 & + & \frac{2}{5}x_6 \end{array}$$

1. Construct the *artificial problem* and the corresponding initial feasible dictionary (as described above).
2. Solve the artificial problem via the Simplex Method.
3. If $w^* = 0$ transform this optimal artificial dictionary into an initial feasible dictionary for the original problem and proceed with the Simplex Method; otherwise, the original problem is *infeasible*.

In phase 1 we solve the artificial problem and in phase 2 we proceed with the original problem if possible.