System of ODE's (Ordinary Differential Equations)

$$\frac{dy_1}{dx} = f_1(y_1, y_2, \ldots\ldots, y_n, x)$$

$$\frac{dy_2}{dx} = f_2(y_1, y_2, \ldots\ldots, y_n, x)$$

$$\vdots$$

$$\frac{dy_n}{dx} = f_n(y_1, y_2, \ldots\ldots, y_n, x)$$

Definitions

y's – dependent variables (e.g. temperature)

x – independent variable (e.g. time)

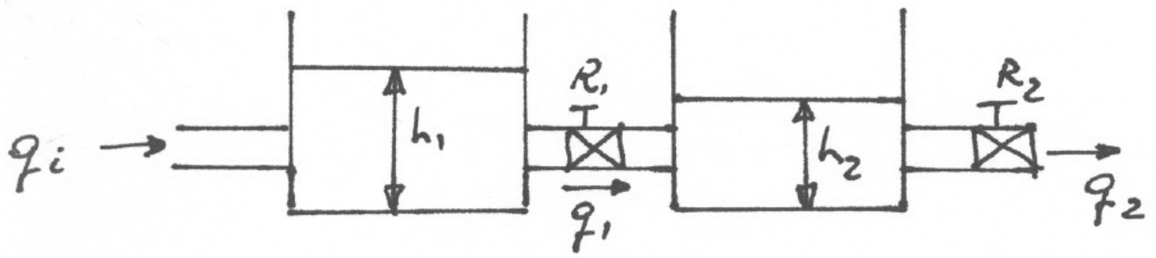
$$\dot{y}_i = \frac{dy_i}{dx}$$

ODE – only 1 independent variable

Solution $\rightarrow$ vector $\bar{y} = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}$

Linearity $\bar{y}_A$ and $\bar{y}_B$ solutions $\Rightarrow \bar{y}_A + \bar{y}_B$ also solution

Motivation Example - 2-Tank Problem



For laminar flow across valves:

$$q_1 = \frac{\Delta P_1}{R_1} \qquad q_2 = \frac{\Delta P_2}{R_2}$$

$R_{1,2}$ - hydraulic resistances

$$\Delta P_1 = P_a + \rho g h_1 - (P_a + \rho g h_2) = \rho g (h_1 - h_2)$$

also $\Delta P_2 = \rho g (h_2 - 0) = \rho g h_2$

<u>Compute</u> $h_1(t)$ and $h_2(t)$

for a given $q_i(t)$

and $h_1(t=0) = h_{10}$ $\quad h_2(t=0) = h_{20}$

Crossectional areas A for both tanks

Mass Balances are needed !!

Accumulation = Input - Output + generation
(- consumption)

Tank # 1

mass

Accumulation : for tank 1 $\dfrac{dm}{dt} = \dfrac{d(\rho A h_1)}{dt}$

Input : q_i

Output : $q_1 = \dfrac{\rho g (h_1 - h_2)}{R_1}$

Generation / Consumption $= 0$ (no evaporation nor condensation)

$\Rightarrow$ Mass Balance for Tank # 1

$$\frac{d(\rho A h_1)}{dt} = q_i - \frac{\rho g\,(h_1 - h_2)}{R_1}$$

if $\rho =$ constant

$$\frac{dh_1}{dt} = \frac{q_i}{\rho A} - \frac{g\,(h_1 - h_2)}{A R_1}$$

In the same way for Tank # 2

$$\frac{dh_2}{dt} = \frac{g(h_1 - h_2)}{A R_1} - \frac{g h_2}{A R_2}$$

Define $R_1' = \dfrac{A R_1}{g}$ $\quad R_2' = \dfrac{A R_2}{g}$ $\quad q_i' = \dfrac{q_i}{\rho A}$

The process is modelled by a system of 2 ODE's:

$$\frac{dh_1}{dt} = q_i' - \frac{(h_1 - h_2)}{R_1'}$$

$$\frac{dh_2}{dt} = \frac{(h_1 - h_2)}{R_1'} - \frac{h_2}{R_2'}$$

Notice that higher order equations can be always converted into 1st order:

e.g. $$\frac{d^2 y_1}{dx^2} = f\left(x, y_1, \frac{dy_1}{dx}\right)$$

Define $y_2 = \frac{dy_1}{dx} \Rightarrow \frac{dy_2}{dx} = \frac{d^2 y_1}{dx^2} = f(x, y_1, y_2)$

Then, the 2nd order equation becomes a set of two 1st order equations:

$$\frac{dy_1}{dx} = y_2$$

$$\frac{dy_2}{dx} = f(x, y_1, y_2)$$

2 Methods of Solution:

1- Systematic Elimination

2- Matrix Algebra

1 - Systematic Elimination

Convert system of n 1^{st} order ODE's into one n^{th} order ODE and solve.

For the 2-tank Problem

$$\frac{dh_1}{dt} = f_i' + \frac{h_2}{R_1'} - \frac{h_1}{R_1'} \qquad (1)$$

$$\frac{dh_2}{dt} = \frac{h_1}{R_1'} - h_2\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \qquad (2)$$

Take derivative of (2) $\Rightarrow$

$$\frac{d^2h_2}{dt^2} = \frac{1}{R_1'}\frac{dh_1}{dt} - \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt} \qquad (3)$$

Substitute (1) into (3)

$$\frac{d^2h_2}{dt^2} = \frac{1}{R_1}\left(q_i' + \frac{h_2}{R_1'} - \frac{h_1}{R_1'}\right) - \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt} \qquad (4)$$

To eliminate h_1 from (4), use equation (2):

$$h_1 = R_1' \frac{dh_2}{dt} + R_1' h_2 \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \qquad (5)$$

Substitute (5) into (4)

$$\frac{d^2h_2}{dt^2} = \frac{1}{R_1'}\left[q_i' + \frac{h_2}{R_1'} - \frac{dh_2}{dt} - h_2\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\right] - \left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt}$$

Or re-arranging

$$\frac{d^2h_2}{dt^2} + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt} + \frac{h_2}{R_1' R_2'} = \frac{q_i'}{R_1'} \qquad (6)$$

This is a 2nd order non-homogeneous ODE

Solve for $q_i' = q_o = \text{const}$

$h_1(t=0) = h_{10} \qquad h_2(t=0) = h_{20}$

Solve $\begin{cases} 1^{st} - \text{homogeneous problem} \\ 2^{nd} - \text{particular problem} \end{cases}$

General solution = homogeneous + particular

Homogeneous Solution

$$\frac{d^2h_2}{dt^2} + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt} + \frac{h_2}{R_1'R_2'} = 0 \qquad (7)$$

Assume, $h_2 = Ae^{\lambda t}$ → substitute into (7)

$$\cancel{A}\lambda^2\cancel{e^{\lambda t}} + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\cancel{A}\lambda\cancel{e^{\lambda t}} + \frac{\cancel{A}\cancel{e^{\lambda t}}}{R_1'R_2'} = 0$$

$$\Rightarrow \lambda^2 + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\lambda + \frac{1}{R_1'R_2'} = 0 \qquad (8)$$

Roots of (8) $\quad \lambda_{1,2} = \dfrac{-\left(\frac{2}{R_1'} + \frac{1}{R_2'}\right) \pm \sqrt{\left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)^2 - \frac{4}{R_1'R_2'}}}{2}$

⇒ homogeneous Solution $\quad h_2 = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}$

Particular Solution

$$\frac{d^2h_2}{dt^2} + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\frac{dh_2}{dt} + \frac{h_2}{R_1'R_2'} = \frac{q_0}{R_1'} \qquad (9)$$

Assume $h_2 = \alpha$ = constant → substitute into (9)

$$\frac{\alpha}{\cancel{R_1'}R_2'} = \frac{q_0}{\cancel{R_1'}} \Rightarrow \alpha = q_0R_2'$$

⇒ General Solution $= h_2 = \underbrace{A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}}_{\text{homogeneous}} + \underbrace{q_0R_2'}_{\text{particular}} \qquad (10)$

Review: 2^{nd} order equations w/ constant coefficients

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = f(t) \qquad (1)$$

General Solution = Homogeneous Solution + Particular Solution

Homogeneous

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = 0 \qquad (2)$$

Assume $y_h = Ae^{\lambda t}$ (3)

Substitute (3) into (2)

$$Ae^{\lambda t}(\lambda^2 + a\lambda + b) = 0 \Rightarrow \lambda^2 + a\lambda + b = 0$$

3 cases (i) $\lambda_{1,2}$ real different

(ii) $\lambda_{1,2}$ complex conjugate ($\lambda = Re \pm Im j$)

(iii) $\lambda_{1,2}$ real and identical

For (i) $y_h = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$

For (ii) $y_h = e^{Re\, t}(A_1 \cos Im\, t + A_2 \sin Im\, t)$

For (iii) $y_h = A_1 e^{\lambda_1 t} + A_2 t e^{\lambda_1 t}$ (2 independent solutions)

<u>Particular Solution</u>: Assume $y_p = \alpha f(t)$ substitute and solve.

Solve for constants A_1 and A_2 from initial conditions →

From (10) $\quad h_2(t=0) = h_{20} = A_1 + A_2 + q_0 R_2' \qquad (11)$

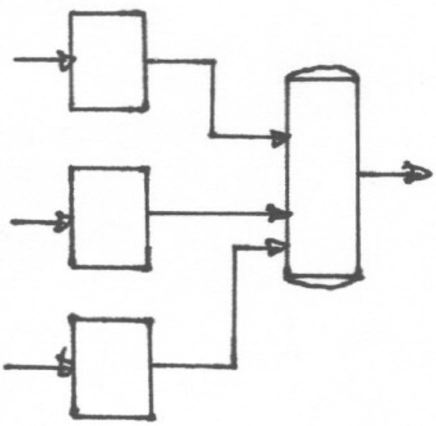
From (2) $\quad \frac{dh_2}{dt}(t=0) = \frac{h_{10}}{R_1'} - h_{20}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \qquad (12)$

From (10) and (12) $\quad A_1 \alpha_1 + A_2 \alpha_2 = \frac{h_{10}}{R_1'} - h_{20}\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \qquad (13)$

(11) and (13) → solve for A_1 and A_2.

Therefore, methodology of elimination is simple but extremely tedious!!

Also, chemical process will generally include a number of units in series or parallel:



elimination will be very difficult!!

<u>Alternative</u>: Matrix Solution

Also suitable for numerical analysis.