Systems of ODE's : Matrix Solution Method

Repeat 2-Tank ODE's equations:

$$\frac{dh_1}{dt} = -\frac{h_1}{R_1'} + \frac{h_2}{R_1'} + q_i' \qquad (1)$$

$$\frac{dh_2}{dt} = \frac{h_1}{R_1'} - h_2\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \qquad (2)$$

In matrix form:

$$\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1'} & \frac{1}{R_1'} \\ \frac{1}{R_1'} & -\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} q_i' \qquad (3)$$

Or in "compact" form:

$$\frac{d\bar{x}}{dt} = \underline{A}\,\bar{x} + \underline{B}\,\bar{u} \qquad \left(\begin{array}{l} \bar{x} - \text{vector} \\ \underline{A} - \text{matrix} \end{array}\right)$$

$$\bar{x} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \qquad \frac{d\bar{x}}{dt} = \begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} \qquad \underline{A} = \begin{bmatrix} -\frac{1}{R_1'} & \frac{1}{R_1'} \\ \frac{1}{R_1'} & -\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (In this case, column vector) special case of matrix

$\bar{u} = q_i'$ (scalar, special case of vector)

For the general case:

$$\frac{d\bar{x}}{dt} = \underline{A}\,\bar{x} + \underline{B}\,\bar{u} \qquad (4)$$

$\bar{x}$ - $n \times 1$ vector , n - # variables/states

$\frac{d\bar{x}}{dt}$ - $n \times 1$ vector

$\underline{A}$ - $n \times n$ matrix

$\underline{B}$ - $n \times m$ matrix

$\bar{u}$ - $m \times 1$ vector, m - # of inputs

Assume $\bar{u} = 0$ (no input)

$$\frac{d\bar{x}}{dt} = \underline{A}\,\bar{x} \qquad \text{homogeneous problem (5)}$$

$$\frac{d\bar{x}}{dt} = \underline{A}\,\bar{x} + \underline{B}\,\bar{u} \qquad \text{non-homogeneous problem (6)}$$

General Solution = Homogeneous + Particular (Non-homogeneous)

Then, substitute initial conditions to solve for constants.

Homogeneous Solution: Matrix Method

Repeat equation (5) $\quad \frac{d\bar{x}}{dt} = \underline{A}\bar{x} \quad (7)$

Assume $\quad \bar{x} = \bar{\alpha} e^{\lambda t} \quad$ $\bar{\alpha}$ - vector, λ - scaler $\quad (8)$

Substitute (8) into (7)

$$\bar{\alpha}\lambda \cancel{e^{\lambda t}} = \underline{A}\bar{\alpha}\cancel{e^{\lambda t}} \quad (9)$$

$$\Rightarrow \quad (\underline{A} - \lambda \underline{I})\bar{\alpha} = 0 \quad (10)$$

Equation (10) is referred to as an "Eigenvalue Problem".

A system of equations $\underline{C}\bar{x} = \bar{b}$ can be generally solved using Cramer's rule:

$$\bar{x} = \underline{C}^{-1}\bar{b} \quad (11)$$

if $\det(\underline{C}) \neq 0 \Rightarrow$ one solution from (11)

$\Rightarrow$ if $\det(\underline{C}) \neq 0$ for problem (10), from (11):

$$\bar{x} = [0] \quad \text{since} \quad \bar{b} = 0$$

$\bar{x} = [0]$ trivial solution of (7). Is there a non-trivial solution?

if $\det(\underline{C}) = 0 \Rightarrow \underline{C}\,\bar{x} = [0]$ linearly dependent equations

$\Rightarrow$ infinite solutions are obtained.

For our case, equation (10) $\rightarrow (\underline{A} - \lambda \underline{I})\,\bar{\alpha} = 0$

$\Rightarrow \det(\underline{A} - \lambda \underline{I}) = 0$ for non-trivial solutions.

<u>Definitions (from Linear Algebra)</u>

<u>Rank of a Matrix</u>:

For the matrix $\underline{C}$ $(n \times n)$

if $\det(\underline{C}) \neq 0 \Rightarrow \text{rank}(\underline{C}) = n \Rightarrow$ rows linearly independent

if $\text{rank}(\underline{C}) = n \rightarrow$ matrix is full rank.

if $\det(\underline{C}) = 0 \Rightarrow \text{rank}(\underline{C}) < n \Rightarrow$ rows linearly dependent

if $\text{rank}(\underline{C}) < n \rightarrow$ matrix is not full rank.

<u>Eigenvalue</u>: is a value of λ for which the solution of $(\underline{A} - \lambda \underline{I})\,\bar{\alpha} = 0$ is non-trivial $(\bar{\alpha} \neq 0) \Rightarrow \det(\underline{A} - \lambda \underline{I}) = 0$

<u>Eigenvector</u>: is the $\bar{\alpha}$ that satisfies $(\underline{A} - \lambda \underline{I})\,\bar{\alpha} = 0$ for each eigenvalue λ.

Example: 2-Tank Problem

Compute Eigenvalues and Eigenvectors for:

from (3) above

$$\underline{A} = \begin{bmatrix} -\frac{1}{R_1'} & \frac{1}{R_1'} \\ \frac{1}{R_1'} & -\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} \quad \left(\begin{matrix} \frac{1}{R_1'} = 2 \\ \frac{1}{R_2'} = 1 \end{matrix}\right)$$

Eigenvalues

$$\det\left\{\begin{bmatrix} -\frac{1}{R_1'} & \frac{1}{R_1'} \\ \frac{1}{R_1'} & -\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} = 0$$

$$\det\left\{\begin{bmatrix} -\frac{1}{R_1'} - \lambda & \frac{1}{R_1'} \\ \frac{1}{R_1'} & -\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) - \lambda \end{bmatrix}\right\} = 0$$

$$\left(-\frac{1}{R_1'} - \lambda\right)\left[-\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right) - \lambda\right] - \frac{1}{R_1'^2} = 0$$

$$\lambda^2 + \left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)\lambda + \frac{1}{R_1'R_2'} = 0$$

eigenvalues

$$\lambda_{1,2} = \frac{-\left(\frac{2}{R_1'} + \frac{1}{R_2'}\right) \pm \sqrt{\left(\frac{2}{R_1'} + \frac{1}{R_2'}\right)^2 - \frac{4}{R_1'R_2'}}}{2}$$

exactly same as in substitution !!

for $\frac{1}{R_1'} = 2 \quad \frac{1}{R_2'} = 1$

$$\lambda_1 = -0.438$$
$$\lambda_2 = -4.562$$

Computation of Eigenvectors $\quad (A - \lambda I)\bar{\alpha} = 0$

$\lambda_1 = -0.438 \quad \rightarrow [\underline{A} - (-0.438)\underline{I}]\bar{\alpha}_1 = 0$

or

$$\underbrace{\left\{\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 0.438 & 0 \\ 0 & 0.438 \end{bmatrix}\right\}}_{\underline{A} - \lambda \underline{I}} \underbrace{\begin{bmatrix} \alpha_{1x} \\ \alpha_{1y} \end{bmatrix}}_{\alpha_1} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

$$\begin{bmatrix} -1.562 & 2 \\ 2 & -2.55 \end{bmatrix}\begin{bmatrix} \alpha_{1x} \\ \alpha_{1y} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{array}{l} -1.562\alpha_{1x} + 2\alpha_{1y} = 0 \\ 2\alpha_{1x} - 2.56\alpha_{1y} = 0 \end{array}\right\}$$

2^{nd} equation is equal to 1^{st} equation when multiplied by $\left(-\frac{1.562}{2}\right)$

$\Rightarrow$ infinite solutions

$\Rightarrow$ select $\alpha_{1y} = C_1$ arbitrary constant

$\Rightarrow -1.562\alpha_{1x} + 2C_1 = 0 \quad \Rightarrow \alpha_{1x} = \frac{2}{1.562}C_1$

$$\Rightarrow \quad \alpha_1 = \begin{bmatrix} \frac{2}{1.562}C_1 \\ C_1 \end{bmatrix}$$

In the same way for $\lambda_2 = -4.56$

$$(\underline{A} - \lambda_2 \underline{I})\bar{\alpha}_2 = 0$$

$$\begin{bmatrix} 2.56 & 2 \\ 2 & 1.56 \end{bmatrix} \begin{pmatrix} \alpha_{2x} \\ \alpha_{2y} \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha_{2y} = C_2$$

$$\Rightarrow 2.56\,\alpha_{2x} + 2\,\alpha_{2y} = 0 \quad \Rightarrow \quad \alpha_{2x} = -\frac{2}{2.56} C_2$$

$$\bar{\alpha}_2 = \begin{bmatrix} -\frac{2}{2.56} C_2 \\ C_2 \end{bmatrix}$$

To summerize

$$\bar{h} = \bar{\alpha} e^{\lambda t}$$

$$\Rightarrow \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{1.562} C_1 \\ C_1 \end{bmatrix} e^{-0.438t} + \begin{bmatrix} -\frac{2}{2.56} C_2 \\ C_2 \end{bmatrix} e^{-4.56t}$$

$$h_1 = \frac{2}{1.562} C_1 e^{-0.438t} - \frac{2}{2.56} C_2 e^{-4.56t}$$

$$h_2 = C_1 e^{-0.438t} + C_2 e^{-4.56t}$$

As expected, equal form to solution with substitution.

C_1 and C_2 has to be solved from initial conditions.
(e.g. for $h_1(t=0) = h_2(t=0) = 0 \Rightarrow C_1 = C_2 = 0$)