Non-homogeneous System (Perticuler Solution)

$$\frac{d\bar{y}}{dt} = \underline{A}\,\bar{y} + \bar{F} \qquad (1)$$

In the previous problem, the homogeneous solution is:

$$\bar{y}_h = \sum_i c_i \bar{x}_i e^{\lambda_i t} = \underline{Y}\,\bar{c} \qquad (2)$$

For example, for the 2-tank problem, :

$$\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} \frac{2}{1.56} c_1 \\ c_1 \end{bmatrix} e^{-0.438t} + \begin{bmatrix} -\frac{2}{2.56} c_2 \\ c_2 \end{bmatrix} e^{-4.56t}$$

$$= \underbrace{\begin{bmatrix} \frac{2}{1.56} e^{-0.438t} & -\frac{2}{2.56} e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix}}_{\underline{Y}} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

The general solution:

$$\bar{y}(t) = \underbrace{\bar{y}_h}_{\text{homogeneous}} + \underbrace{\bar{y}_p}_{\text{perticuler or non-homogeneous}} = \underline{Y}\,\bar{c} + \bar{y}_p \qquad (3)$$

Solve $\bar{y}_p$ with the method of veriation of parameters:

Assume a solution

$$y_p(t) = \underline{Y}\,\bar{z} \qquad (4)$$

where $\underline{Y}$ is from (2) (homogeneous solution) and $\bar{z}$ is a vector of functions to be found in the sequel.

Finding $\bar{z}$

Substitute (4) into (1) :

$$\frac{d(\underline{Y}\bar{z})}{dt} = \underline{A}\,\underline{Y}\,\bar{z} + \bar{F}$$

$$\Rightarrow \quad \frac{d\underline{Y}}{dt}\bar{z} + \underline{Y}\frac{d\bar{z}}{dt} = \underline{A}\,\underline{Y}\,\bar{z} + \bar{F}$$

$$\left(\frac{d\underline{Y}}{dt} - \underline{A}\,\underline{Y}\right)\bar{z} + \underline{Y}\frac{d\bar{z}}{dt} = \bar{F} \qquad (5)$$

But, $\frac{d\underline{Y}}{dt} - \underline{A}\,\underline{Y} = 0$, because it is the homogeneous solution !!

$$\Rightarrow \quad \underline{Y}\frac{d\bar{z}}{dt} = \bar{F} \quad \Rightarrow \quad \underline{Y}^{-1}\underline{Y}\frac{d\bar{z}}{dt} = \underline{Y}^{-1}\bar{F}$$

$$\Rightarrow \quad \bar{z} = \int_0^t \underline{Y}^{-1}\bar{F}\,dt \qquad (6)$$

From (4) and (6) $\quad y_p = \underline{Y}\int_0^t \underline{Y}^{-1}\bar{F}\,dt \qquad (7)$

Review:

Inverse of a Matrix

$$\underline{A}^{-1} = \frac{1}{\det \underline{A}} \left[A_{jk} \right]^T$$

A_{jk} is the cofactor of the elements in A calculated as follows:

e.g for 3x3 matrix $\quad \underline{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The minors of A are defined as follows:

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

in general $\quad M_{ij} = \det \left[\begin{array}{l} \text{leftover after taking} \\ \text{i-th row and j-th column} \end{array} \right]$

The cofactors are:

$$A_{11} = M_{11} \qquad A_{12} = -M_{12} \qquad A_{13} = M_{13}$$

$$A_{21} = -M_{21} \qquad A_{22} = M_{22} \qquad A_{23} = -M_{23}$$

$$A_{31} = M_{31} \qquad A_{32} = -M_{32} \qquad A_{33} = M_{33}$$

example: for 2×2 $\quad \underline{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \underline{A}^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\det \underline{A}}$

Example:

Complete the 2-tank problems with $q_i = 1$

$$\begin{bmatrix} \frac{dh_1}{dt} \\ \frac{dh_2}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 2 \\ 2 & -3 \end{bmatrix}}_{\underline{A}} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{F}$$

Find the general solution to the problem

$$\bar{y}_P = \underline{Y}\bar{z} \qquad \underline{Y} = \begin{bmatrix} \frac{2}{1.56} e^{-0.438t} & -\frac{2}{2.56} e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix}$$

$$\Rightarrow \bar{y}_P = \underline{Y} \int_0^t \underline{Y}^{-1} F \, dt =$$

$$= \begin{bmatrix} \frac{2}{1.56} e^{-0.438t} & -\frac{2}{2.56} e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix} \int_0^t \frac{\begin{bmatrix} e^{-4.56t} & \frac{2}{2.56} e^{-4.56t} \\ -e^{-0.438t} & \frac{2}{1.56} e^{-0.438t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\left(\frac{2}{1.56} e^{-5t} + \frac{2}{2.56} e^{-5t} \right)} dt$$

$$= \begin{bmatrix} \frac{2}{1.56} e^{-0.438t} & -\frac{2}{2.56} e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix} \int_0^t \frac{\begin{bmatrix} e^{-4.56t} \\ -e^{-0.438t} \end{bmatrix}}{2.06 e^{-5t}} dt$$

$$= \frac{1}{2.06} \begin{bmatrix} \frac{2}{1.56} e^{-0.438t} & -\frac{2}{2.56} e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix} \int_0^t \begin{pmatrix} e^{+0.438t} \\ -e^{+4.56t} \end{pmatrix} dt =$$

$$= \frac{1}{2.06}\begin{bmatrix} \frac{2}{1.56}e^{-0.438t} & -\frac{2}{2.56}e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix}\begin{bmatrix} e^{0.438t}/0.438 \\ -e^{4.56}/4.56 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2.06}\cdot\frac{2}{1.56}\frac{1}{0.438} + \frac{1}{2.06}\cdot\frac{2}{2.56}\cdot\frac{1}{4.56} \\ \frac{1}{2.06}\cdot\frac{1}{0.438} - \frac{1}{2.06}\cdot\frac{1}{4.56} \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

As expected since $\bar{F}$ is a constant vector the particular solution is also a constant vector.

The general solution is:

$$\bar{y} = \bar{y}_h + \bar{y}_p = \underbrace{\begin{bmatrix} \frac{2}{1.56}e^{-0.438t} & -\frac{2}{2.56}e^{-4.56t} \\ e^{-0.438t} & e^{-4.56t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}}_{\bar{y}_h} + \underbrace{\begin{bmatrix} 1.5 \\ 1 \end{bmatrix}}_{\bar{y}_p}$$

C_1 and C_2 are computed from:

$$h_1(t=0) = h_{10} \qquad h_2(t=0) = h_{20}$$