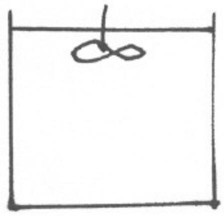
Partial Differential Equations (PDE's)

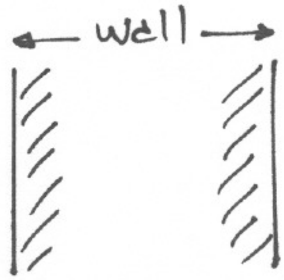


$x \quad T \neq f(x)$

Lumped System

$\Downarrow$

ODE



$x \quad T = f(x)$

Distributed System

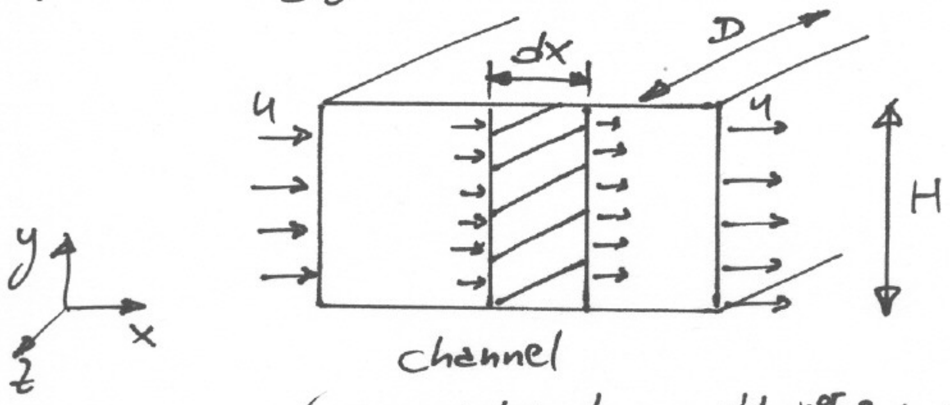
$\Downarrow$

PDE

For distributed Systems write:

Mass
Energy
Momentum
} microscopic balances

Microscopic Energy Balance (1-dimension)



channel
(no gradients in the y or z direction)

For the infinitesimal volume (width dx):

accumulation of energy = energy in − energy out + generation − consumption

generation (by reaction, phase change etc)

Consumption (by reaction, phase change, work etc)

Assume for this case generation/consumption = 0
Assume ρ and C_p are constant
$\Rightarrow$

accumulation of energy = energy in − energy out (1)

energy transported by 2 mechanisms $\left\{ \begin{array}{l} \text{conduction} \\ \text{convection} \end{array} \right.$

$$\text{Accumulation} = \rho C_p \overbrace{dx\,H\,D}^{\text{volume}} \frac{\partial T}{\partial t} \qquad (2)$$

$$\text{Energy In} = \underbrace{\rho C_p\, u . T . \underbrace{HD}_{\text{crossection area}}}_{\substack{\text{heat convected in} \\ \text{(enthalpy)}}} + \underbrace{\left(-k \frac{\partial T}{\partial x}\right) HD}_{\substack{\text{heat conducted in} \\ \text{(Fourier law)}}} \qquad (3)$$

$$\text{Energy out} = \underbrace{\rho C_p u T H D + \frac{\partial}{\partial x}(\rho C_p u T H D)\,dx}_{\text{heat convected out}}$$

$$+ \underbrace{\left(-k\frac{\partial T}{\partial x}\right) HD + \frac{\partial}{\partial x}\left[\left(-k\frac{\partial T}{\partial x}\right) HD\right] dx}_{\text{heat conducted out}} \qquad (4)$$

Substitute (2), (3) and (4) into (1) :

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial}{\partial x}(\rho C_p u T) + \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \qquad (5)$$

If ρ, C_p, u and k are constant

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p u \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2} \qquad (6)$$

In 3 dimensions:

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

(7)

With reaction:

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + r\left(-\Delta H\right)$$

r - reaction rate ΔH - heat of reaction
(negative value - exothermic reaction)

Mass Balance Equation (very similar)

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = D\left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} + \frac{\partial^2 c}{\partial z^2}\right) - r$$

(r - positive = consumption of species)

Continuity Equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

Momentum Equations (Navier-Stokes)

Classification of PDE's

For the general PDE

$$P\frac{\partial^2 z}{\partial x^2} + Q\frac{\partial^2 z}{\partial x \partial y} + R\frac{\partial^2 z}{\partial y^2} = \delta$$

δ may be a function of $x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

In analogy with the equation:

$$Px^2 + 2Q\,xy + R\,y^2 = S$$

Define $\Delta = Q^2 - 4PR$

$\Delta < 0$ (e.g $Q = 0$ $P, R > 0$) elliptic equation

$\Delta = 0$ (e.g $Q = 0, P = 0$) parabolic equation

$\Delta > 0$ (e.g. $R \cdot P = 0$) hyperbolic equation

Examples

Transient Heat Transfer - 1D $\quad \frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2} \quad S = \frac{\partial T}{\partial t}, R = D, P, Q = 0$

$\Delta = 0 \Rightarrow$ parabolic

Steady state, Mass Transfer - 2D $\quad \frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0$

$R = P = 1 \quad Q = 0 \Rightarrow \Delta < 0 \Rightarrow$ elliptic

Transient, String Wave Equation $\frac{\partial^2 u}{\partial t^2} = \rho\frac{\partial^2 u}{\partial x^2} \rightarrow$ hyperbolic

Is there a fundamental difference between the equations?

Look at the homogeneous equation:

$$P\frac{\partial^2 z}{\partial x^2} + Q\frac{\partial^2 z}{\partial x \partial y} + R\frac{\partial^2 z}{\partial y^2} = 0 \qquad (1)$$

Assume a solution $\quad z = f(y + \lambda x) \qquad (2)$

f - is an unknown function

Define a new variable $\quad \eta = y + \lambda x \qquad (3)$

From (2) and (3)

$$\frac{\partial^2 z}{\partial x^2} = \lambda^2 \frac{\partial^2 f}{\partial \eta^2} \qquad \frac{\partial^2 z}{\partial x \partial y} = \lambda \frac{\partial^2 f}{\partial \eta^2} \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial \eta^2} \qquad (4)$$

Substitute (4) into (1)

$$P\lambda^2 \cancel{\frac{\partial^2 f}{\partial \eta^2}} + Q\lambda \cancel{\frac{\partial^2 f}{\partial \eta^2}} + R\cancel{\frac{\partial^2 f}{\partial \eta^2}} = 0$$

$$\Rightarrow \quad P\lambda^2 + Q\lambda + R = 0 \qquad (5)$$

$$\lambda_{1,2} = \frac{-Q \pm \sqrt{Q^2 - 4RP}}{2P} \qquad (6)$$

Thus, the solution depends on $\Delta = Q^2 - 4RP$

Since there are 2-solutions

$$f_1(\quad + \lambda_1 x) \qquad f_2(y + \lambda_2 x)$$

Solution (linear system) by superposition:

$$z = f_1(y+\lambda_1 x) + f_2(y+\lambda_2 x) \tag{7}$$

if $\Delta > 0$, hyperbolic equation with 2 real roots

if $\Delta = 0$, parabolic equation with equal roots

$$\Rightarrow \quad z = f_1(y+\lambda_1 x) + \underbrace{x\, f_{11}(y+\lambda_1 x)}_{\substack{2^{nd}\text{ independent solution}\\ \text{(proof by substitution)}}}$$

if $\Delta < 0$, elliptic equation with 2 complex roots

$$\Rightarrow \quad z = f_1(y + i\lambda_1 x) + f_2(y - i\lambda_1 x) \tag{8}$$

Thus, the solutions are different dependent on the value of λ.

The question is what are f_1, f_2 ??

This requires solution of the PDE's

Solution Methods for PDE's

Analytical
- 1- Seperation of Variebles ✓
- 2- Laplace Transform ✓
- 3- Similarity

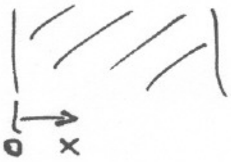
Approximate
- 1- Perturbation Techniques

Numerical
- 1- Finite Differences ✓
- 2- Finite Elements
- 3- Orthogonal Collocations

Boundary Conditions

1- Function Specified at Boundary

e.g.

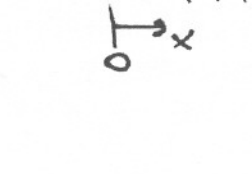


@ $x=0$ $T=T_1$

2- Derivative specified at Boundary

@ $x=0$ $\frac{\partial c}{\partial x}=0$

3- Mixed Condition at Boundary $-k\frac{\partial T}{\partial x} = h\left(T\big|_{x=0} - T_\infty\right)$



h - heat transfer coeff.