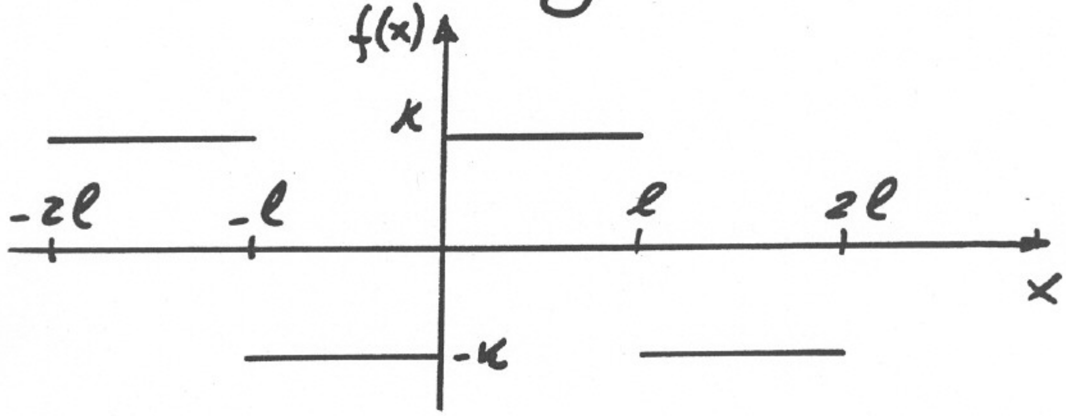
Example : Fourier Series

Apply the Fourier Series Method to represent the following function:



"Periodic Square Wave"

$$a_0 = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(x)\,dx = \frac{1}{2\ell}\int_{-\ell}^{0}(-K\,dx) + \frac{1}{2\ell}\int_{0}^{\ell} K\,dx =$$

$$= -\frac{K}{2\ell}\ell + \frac{K}{2\ell}\ell = 0$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx = \frac{1}{\ell}\int_{-\ell}^{0} -K\cos\frac{n\pi x}{\ell}\,dx$$

$$+\frac{1}{\ell}\int_{0}^{\ell} K\cos\frac{n\pi x}{\ell}\,dx$$

$$a_n = -\frac{K}{\ell}\,\frac{\ell}{n\pi}\,\sin\frac{n\pi x}{\ell}\Big|_{-\ell}^{0} + \frac{K}{\ell}\,\frac{\ell}{n\pi}\,\sin\frac{n\pi x}{\ell}\Big|_{0}^{\ell}$$

$$= 0$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,\sin\frac{n\pi x}{\ell}\,dx =$$

$$= \frac{1}{\ell}\int_{-\ell}^{\ell}(-K)\,\sin\frac{n\pi x}{\ell}\,dx + \frac{1}{\ell}\int_{0}^{\ell} K\sin\frac{n\pi x}{\ell}\,dx$$

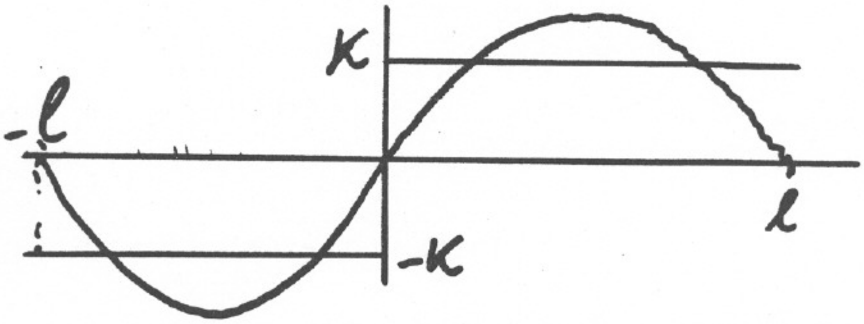
$$= \frac{K}{\ell}\,\frac{\ell}{n\pi}\,\cos\frac{n\pi x}{\ell}\Big|_{-\ell}^{0} - \frac{K}{\ell}\,\frac{\ell}{n\pi}\,\cos\frac{n\pi x}{\ell}\Big|_{0}^{\ell}$$

$$= \frac{K}{n\pi}\left(1-\cos n\pi\right) - \frac{K}{n\pi}\left(\cos n\pi - 1\right)$$

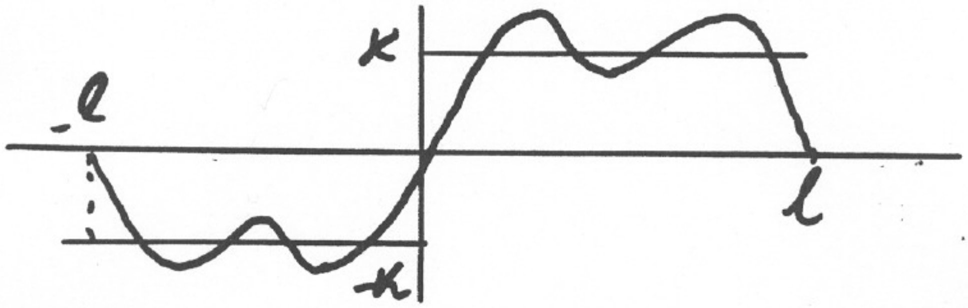
$$= \frac{2K}{n\pi} - \frac{2K}{n\pi}\cos n\pi$$

$$\Rightarrow \quad b_n \quad \begin{array}{ll} 0 & \text{for } n = 2, 4, \ldots \\ \frac{4K}{n\pi} & \text{for } n = 1, 3, 5 \ldots \end{array}$$

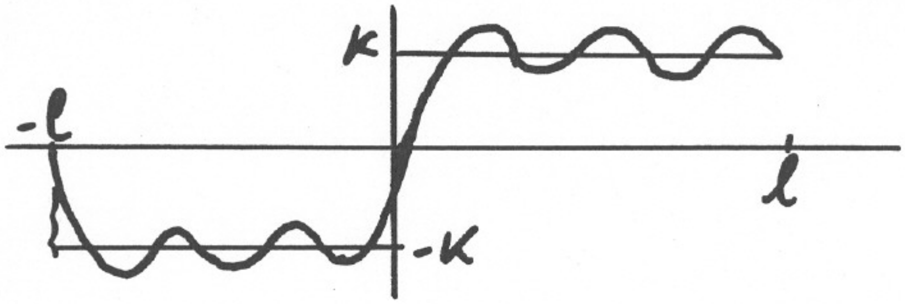
$$f(x) = \frac{4K}{\pi} \sin \frac{\pi x}{l} + \frac{4K}{3\pi} \sin \frac{3\pi x}{l} + \frac{4K}{5\pi} \sin \frac{5\pi x}{l} + \dots$$



1st term



Sum of first 2 terms



Sum of first 3 terms

Simplification of the method

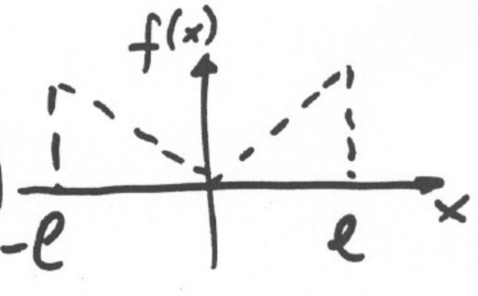
In the square wave example a_0 and a_n are zero.

Could we predict these results without performing the corresponding integrations?

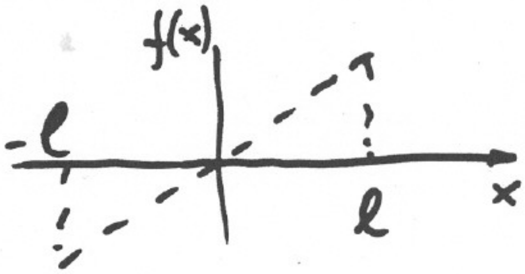
yes

It depends on the function $f(x)$ being ODD or EVEN

EVEN $\left(\int_{-\ell}^{\ell} f(x)\,dx = 2\int_0^{\ell} f\,dx \right)$



ODD $\left(\int_{-\ell}^{\ell} f(x)\,dx = 0 \right)$



The product

odd function × even function =

= odd function

Now, look at the Fourier coefficients:

Assume <u>f is even</u>

sine → odd

$$b_n = \int_{-l}^{l} \underbrace{\underbrace{f}_{\text{even}} \underbrace{\sin \frac{n\pi x}{l}}_{\text{odd}}}_{\text{odd}} dx = 0$$

$a_n \neq 0$

⇒ f will be represented by series of cosines

Assume <u>f is odd</u> (e.g. square wave)

$$a_n = \int_{-l}^{l} \underbrace{\underbrace{f}_{\text{odd}} \underbrace{\cos \frac{n\pi x}{l}}_{\text{even}}}_{\text{odd}} dx = 0$$

$b_n \neq 0$

⇒ f represented by series of sines.

Orthogonality of Functions

In the previous section (Fourier Series) we have used several times the following property:

$$\int_{-\pi}^{\pi} \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}\,dx = \begin{cases} n=m & \text{constant} \neq 0 \\ n\neq m & 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos\frac{n\pi x}{L}\,\cos\frac{m\pi x}{L}\,dx = \begin{cases} n=m & \text{constant} \neq 0 \\ n\neq m & 0 \end{cases}$$

Then $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal

In the same way $\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal.

Proof:

$$\int_{-\pi}^{\pi} \sin\frac{n\pi x}{L}\,\sin\frac{m\pi x}{L}\,dx = \int_{-\pi}^{\pi} \frac{1}{2}\left[\cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\right]dx = \begin{cases} \pi & \text{if } n=m \\ 0 & \text{if } n\neq m \end{cases}$$

In general two functions $P_n(x)$ and $P_m(x)$ are said to be orthogonal iff:

$$\int_a^b r(x)\,P_n(x)\,P_m(x)\,dx = \begin{cases} 0 & \text{if } m\neq n \\ \neq 0 & \text{if } m=n \end{cases}$$

where $r(x)$ is a weighting function.

Completion of the 1-D Rod Heat Transfer Problem

From (11) before, the general solution is:

$$T(x,t) = \sum_{n=0}^{\infty} E_n e^{-p_n^2 c^2 t} \sin \frac{n\pi}{L} x \qquad (11)$$

after substitution of initial condition:

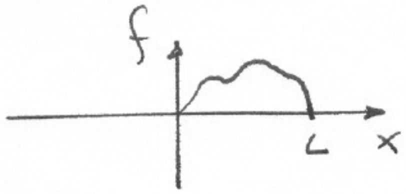
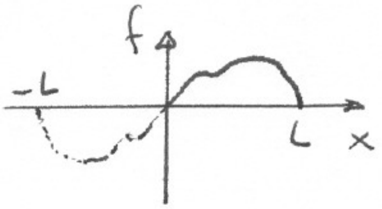
$$T(x,0) = f(x) = \sum_{n=0}^{\infty} E_n \sin \frac{n\pi}{L} x \qquad (12)$$

The idea is to represent $f(x)$ as a series of sines and cosines (Fourier Series) and then equate coefficients in (12)

So, expanding $f(x)$ as a Fourier Series, in general:

$$f(x) = a_0 + \sum_{n=0}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right) \quad (13)$$

To be able to satisfy (12) $a_0, a_n = 0$, $\Rightarrow$

If  expand as odd function in $-L < x < L$ 

$\Rightarrow$ to obtain a series of sines only.

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx \qquad (14)$$

From (12)

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L} = \sum_{n=0}^{\infty} E_n \sin\frac{n\pi x}{L} \qquad (15)$$

Thus, from (14) and (15) $E_n = b_n$

Thus, the solution is from (11) and (14) and (15)

$$T(x,t) = \sum_{n=0}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx\right)\sin\frac{n\pi x}{L} \qquad (16)$$

Simple Example:

Initial Condition $f(x) = \alpha \sin\frac{\pi x}{L} = T(x, t=0)$

From (14)

$$E_n = b_n = \frac{2}{L}\int_0^L \alpha \sin\frac{\pi x}{L} \sin\frac{n\pi x}{L}\, dx \qquad (17)$$

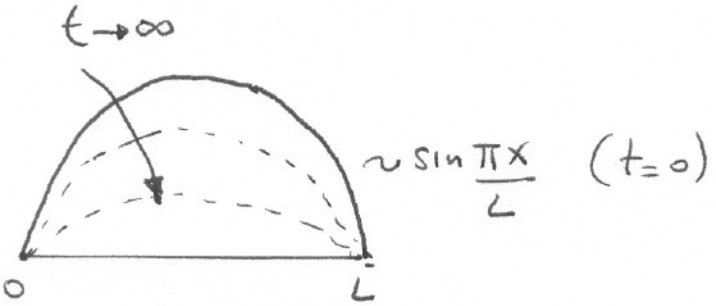
Easy to show that

$E_n = 0 \qquad n \neq 1 \qquad$ because $\int \sin\frac{\pi x}{L} \sin\frac{n\pi x}{L} = 0 \quad n \neq 1$

$n = 1 \qquad E_n = \frac{2\alpha}{L}\int_0^L \sin^2\frac{\pi x}{L}\,dx = \frac{2\alpha}{L}\int_0^L\left(\frac{1}{2} - \frac{1}{2}\cos\frac{2\pi x}{L}\right)dx = \alpha$

Then

$$T(x,t) = \alpha e^{-\frac{n^2\pi^2}{L^2}c^2 t} \sin\frac{\pi x}{L}$$



The solution can be easily obtained by inspection

From (12) above

$$\alpha \sin\frac{\pi x}{L} = \sum_{n=0}^{\infty} E_n \sin\frac{n\pi x}{L}$$

$$\Rightarrow \quad E_1 = \alpha$$

$$E_2 = E_3 = \cdots = 0$$