On the Shannon Capacity of the Fading Three-Node Network

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Consider a broadcast scenario in a three node network, where the source node broadcasts information to both the relay and destination nodes. Let γ_{sd}, γ_{sr}, and γ_{rd} respectively denote the instantaneous channel SNR of each of the three wireless channels, which are governed by the Nakagami m-distribution:

$$f_{\Gamma_x}(\gamma_x) = \left(\frac{m_x}{\overline{\gamma}_x}\right)^{m_x} \frac{\gamma^{m_x-1}}{\Gamma(m_x)} \exp\left(-\frac{m}{\overline{\gamma}_x}\gamma_x\right), \tag{1}$$

where $x \in \{sd, sr, rd\}$.

Now consider two cases: (i) $\gamma_{sr} \le \gamma_{sd}$; (ii) $\gamma_{sr} > \gamma_{sd}$. In case (i), the channel capacity of the *s-r* channel is less than the that of the *s-d* channel. Hence, more information of the source node broadcast is received at the destination node than the relay and the relay fails to provide any additional information. In case (ii), however, the opposite is true and the relay node receives more information than the destination node from the same broadcast. The optimal strategy is for the relay to forward the additional information it receives along the *r-d* channel to the destination, but depending on the size of such information, the transmission may be limited by the capacity of the *r-d* channel.

Denote C^{sd}, C^{sr}, and C^{rd} as the capacity of the respective channels. The effective capacity of the three-node network is thus expressed as:

$$R_0 = \begin{cases} C^{sd} & \text{if } \gamma_{sr} \le \gamma_{sd}; \\ C^{sd} + \min\left\{C^{sr} - C^{sd}, C^{rd}\right\} & \text{if } \gamma_{sr} > \gamma_{sd}. \end{cases} \tag{2}$$

Eq. (2) is extendable to two layers using SPC. Suppose we employ SPC modulation with power allocation parameters β_1 and β_2 at the source and relay nodes, respectively. With SIC-based decoding, the capacities of each channel can now be separated into two layers such that $C^{sd} = C_1^{sd} + C_2^{sd}$, $C^{sr} = C_1^{sr} + C_2^{sr}$, and $C^{rd} = C_1^{rd} + C_2^{rd}$, where

$$C_1^x = \begin{cases} \log\left[1 + \frac{\beta_1\gamma_x}{1+(1-\beta_1)\gamma_x}\right] & \text{if } x = sd, sr \\ \log\left[1 + \frac{\beta_2\gamma_x}{1+(1-\beta_2)\gamma_x}\right] & \text{if } x = rd \end{cases} \tag{3}$$

$$C_2^x = \begin{cases} \log\left[1 + (1-\beta_1)\gamma_x\right] & \text{if } x = sd, sr \\ \log\left[1 + (1-\beta_2)\gamma_x\right] & \text{if } x = rd \end{cases} \tag{4}$$

Due to the dependency of layer 2 on layer 1 in SPC, however, the total achieved capacity along each channel depends on the condition of each channel. Define $\gamma_{th,1}(\beta_1)$ as the SNR threshold such that only layer 1 of the SPC broadcast is decodable along channel $x \in \{sd, sr\}$ if $\gamma_x \le \gamma_{th,1}(\beta_1)$, while both layers are decodable if $\gamma_x > \gamma_{th,1}(\beta_1)$. Similarly define $\gamma_{th,2}(\beta_2)$ for $x = rd$, such that both layers are only decodable if $\gamma_{rd} > \gamma_{th,2}(\beta_2)$. The SNR threshold $\gamma_{th,1}(\beta_1)$ and $\gamma_{th,2}(\beta_2)$ defined in this manner captures the outage probability effects of channel capacity through the consideration of a maximum error threshold ϵ_{th} that must be satisfied in order to achieve reliable decoding in a practical system. Since the error probabilities of each layer is a function of the power allocation parameters, so must the SNR thresholds.

The number of end-to-end decodable layers at the destination hence depends on the SNR of each channel in relation to $\gamma_{th,1}(\beta_1)$ and $\gamma_{th,2}(\beta_2)$. Only layer 1 is decodable under two scenarios:

$$\mathcal{L}_{11} = \{\gamma_{sd} \le \gamma_{th,1}, \gamma_{sr} \le \gamma_{sd}\}; \tag{5}$$

$$\mathcal{L}_{12} = \{\gamma_{sd} \le \gamma_{th,1}, \gamma_{sr} \le \gamma_{th,1}, \gamma_{sr} > \gamma_{sd}\} \cup \{\gamma_{sd} \le \gamma_{th,1}, \gamma_{sr} > \gamma_{th,1}, \gamma_{rd} \le \gamma_{th,2}, \gamma_{sr} > \gamma_{sd}\}. \tag{6}$$

Both layers are decodable under the scenarios:

$$\mathcal{L}_{21} = \{\gamma_{sd} > \gamma_{th,1}, \gamma_{sr} \leq \gamma_{sd}\}; \tag{7}$$

$$\mathcal{L}_{22} = \{\gamma_{sd} > \gamma_{th,1}, \gamma_{sr} > \gamma_{sd}\} \cup \{\gamma_{sd} \leq \gamma_{th,1}, \gamma_{sr} > \gamma_{th,1}, \gamma_{rd} > \gamma_{th,2}, \gamma_{sr} > \gamma_{sd}\}. \tag{8}$$

Define $R' = C_1^{sr} - C_1^{sd}$ and $R'' = C_2^{sr} - C_2^{sd}$. If only layer 1 is decodable, the effective rate received at the destination node is expressed as:

$$R_1 = \begin{cases} C_1^{sd} & \text{if } \gamma_{sr} \leq \gamma_{sd}, \\ C_1^{sd} + \min\left\{R', C_1^{rd}\right\} & \text{if } \gamma_{sr} > \gamma_{sd}. \end{cases} \tag{9}$$

However, if both layers are decodable, the effective rate becomes $R_1 + R_2$, where R_1 is as above and R_2 is expressed as:

$$R_2 = \begin{cases} C_2^{sd} & \text{if } \gamma_{sr} \leq \gamma_{sd}, \\ C_2^{sd} + \min\left\{R'', C_2^{rd}\right\} & \text{if } \gamma_{sr} > \gamma_{sd}. \end{cases} \tag{10}$$

Finally, define four additional events based on the limiting capacity of the *r-d* channel as follows:

$$\mathcal{S}_1 = \{C_1^{rd} < R', C_2^{rd} < R'', \gamma_{sr} > \gamma_{sd}\}; \tag{11}$$

$$\mathcal{S}_2 = \{C_1^{rd} > R', C_2^{rd} > R'', \gamma_{sr} > \gamma_{sd}\}; \tag{12}$$

$$\mathcal{S}_3 = \{C_1^{rd} < R', C_2^{rd} > R'', \gamma_{sr} > \gamma_{sd}\}; \tag{13}$$

$$\mathcal{S}_4 = \{C_1^{rd} > R', C_2^{rd} < R'', \gamma_{sr} > \gamma_{sd}\}. \tag{14}$$

The expected effective rate at the destination node is hence expressed as follows:

$$\begin{aligned} \mathrm{E}[R_{eff}] &= \mathrm{E}[R_1 \mid \mathcal{L}_{11}]\Pr\{\mathcal{L}_{11}\} + \mathrm{E}[R_1 + R_2 \mid \mathcal{L}_{21}]\Pr\{\mathcal{L}_{21}\} \\ &\quad + \sum_{i=1}^{4}\mathrm{E}[R_1 \mid \mathcal{S}_i\mathcal{L}_{12}]\Pr\{\mathcal{S}_i\mathcal{L}_{12}\} + \sum_{i=1}^{4}\mathrm{E}[R_1 + R_2 \mid \mathcal{S}_i\mathcal{L}_{22}]\Pr\{\mathcal{S}_i\mathcal{L}_{22}\} \qquad (15) \\ &= \mathrm{E}[C_1^{sd} \mid \mathcal{L}_{11}]\Pr\{\mathcal{L}_{11}\} + \mathrm{E}[C^{sd} \mid \mathcal{L}_{21}]\Pr\{\mathcal{L}_{21}\} \\ &\quad + \mathrm{E}[C_1^{sd} + C_1^{rd} \mid \mathcal{S}_1\mathcal{L}_{12}]\Pr\{\mathcal{S}_1\mathcal{L}_{12}\} + \mathrm{E}[C^{sd} + C^{rd} \mid \mathcal{S}_1\mathcal{L}_{22}]\Pr\{\mathcal{S}_1\mathcal{L}_{22}\} \\ &\quad + \mathrm{E}[C_1^{sr} \mid \mathcal{S}_2\mathcal{L}_{12}]\Pr\{\mathcal{S}_2\mathcal{L}_{12}\} + \mathrm{E}[C^{sr} \mid \mathcal{S}_2\mathcal{L}_{22}]\Pr\{\mathcal{S}_2\mathcal{L}_{22}\} \\ &\quad + \mathrm{E}[C_1^{sd} + C_1^{rd} \mid \mathcal{S}_3\mathcal{L}_{12}]\Pr\{\mathcal{S}_3\mathcal{L}_{12}\} + \mathrm{E}[C_1^{sd} + C_1^{rd} + C_2^{sr} \mid \mathcal{S}_3\mathcal{L}_{22}]\Pr\{\mathcal{S}_3\mathcal{L}_{22}\} \\ &\quad + \mathrm{E}[C_1^{sr} \mid \mathcal{S}_4\mathcal{L}_{12}]\Pr\{\mathcal{S}_4\mathcal{L}_{12}\} + \mathrm{E}[C_1^{sr} + C_2^{sd} + C_2^{rd} \mid \mathcal{S}_4\mathcal{L}_{22}]\Pr\{\mathcal{S}_4\mathcal{L}_{22}\}. \qquad (16) \end{aligned}$$

Given the channel parameters for each of the three fading channels, each term in Eq. (15) can be evaluated as follows:

$$\mathrm{E}[g(\gamma_{sd}, \gamma_{sr}, \gamma_{rd}) \mid \mathcal{K}]\Pr\{\mathcal{K}\} = \iiint\limits_{\mathcal{K}} g(\gamma_{sd}, \gamma_{sr}, \gamma_{rd}) f_{\Gamma_{sd}} f_{\Gamma_{sr}} f_{\Gamma_{rd}} \mathrm{d}\gamma_{sd}\mathrm{d}\gamma_{sr}\mathrm{d}\gamma_{rd}. \tag{17}$$