Laplace Transform: Definition and Region of Convergence

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Some slides included are extracted from lecture notes from MIT open courseware
http://ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-003Fall-2003/CourseHome/

Why do we need another transform?

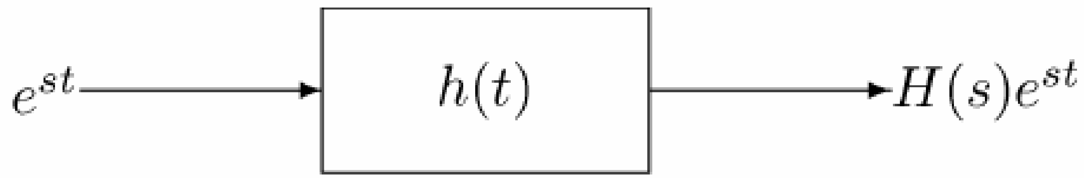
- Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)|dt = \infty$$

- Laplace Transform can be viewed as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems (including unstable systems!)

Eigen Function of LTI System

- e^{st} is an eigenfunction of *any* LTI system
  - $s = \sigma + j\omega$ can be complex in general



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \quad \text{(assuming this converges)}$$

  - Show on the board
  - H(s) is the Laplace transform of h(t)!

The (Bilateral) Laplace Transform

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a *complex* variable

Relation with Fourier Transform

(1) $X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}] e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\}$

(2) A critical issue in dealing with Laplace transform is convergence:
— $X(s)$ generally exists only for *some* values of s, located in what is called the *region of convergence* (ROC)

$$\text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{\substack{\text{Depends} \\ \text{only on } \sigma \\ \text{not on } \omega}} dt < \infty\}$$

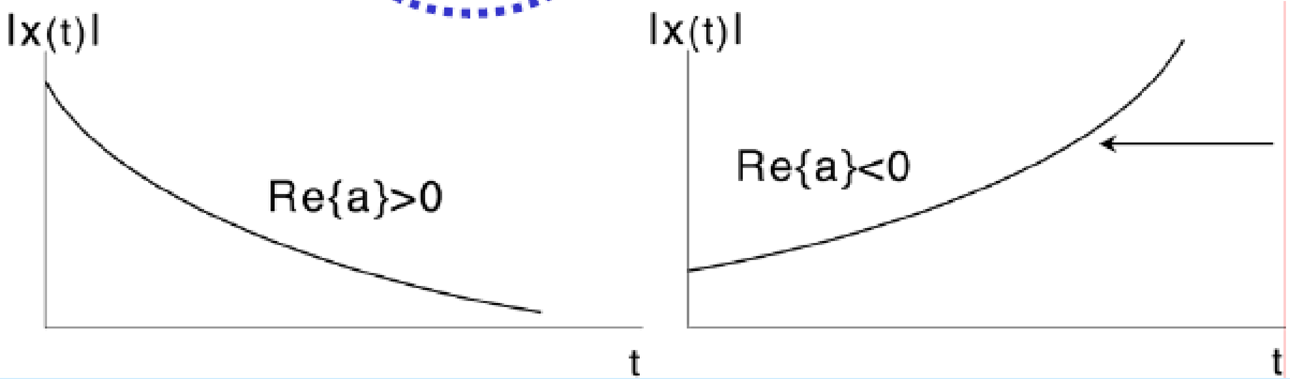
absolute integrability condition

(3) If $s = j\omega$ is in the ROC (i.e. $\sigma = 0$), then

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

Example 1

$x_1(t) = e^{-at}u(t)$ (a – an arbitrary real or complex number)



Unstable:

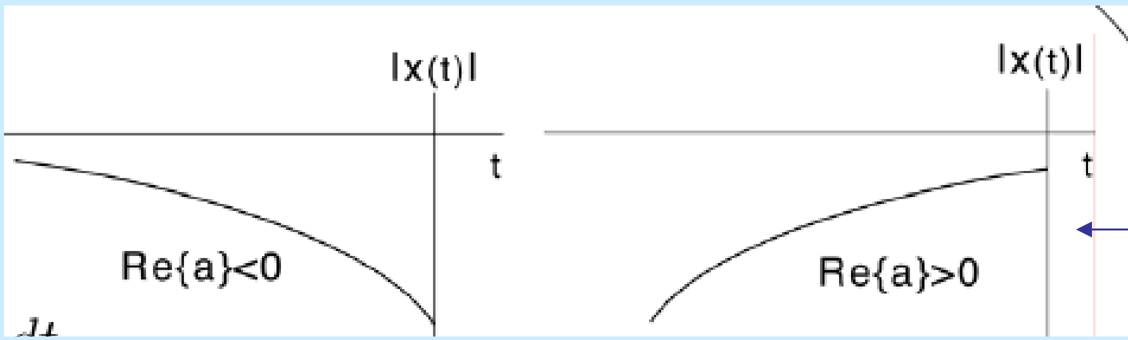
- no *Fourier Transform*
- but *Laplace Transform* exists

$$X_1(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} > -\Re e\{a\}}_{\text{ROC}}$$

- Derive result on board, sketch ROC for both $a>0$ and $a<0$

Example 2

$$x_2(t) = -e^{-at}u(-t)$$



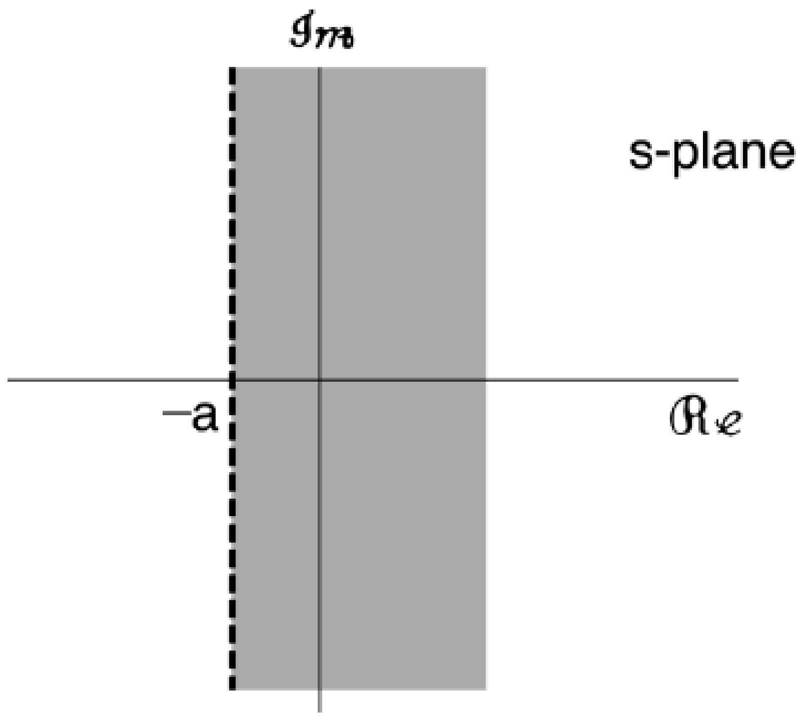
Unstable:
- no *Fourier Transform*
- but *Laplace Transform* exists

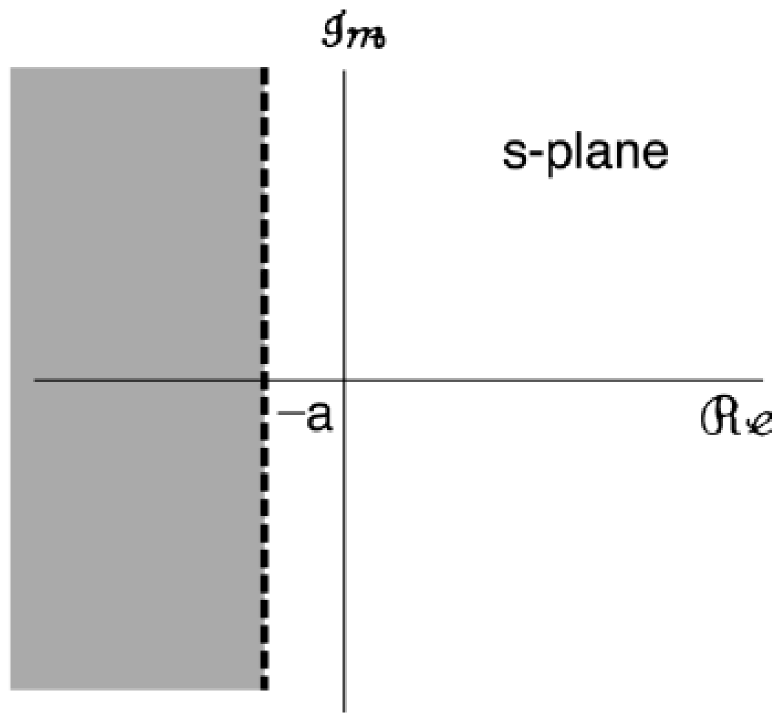
$$X_2(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} < -\Re e\{a\}}_{\text{ROC}}$$

Same as $X_1(s)$, but different ROC

- Derive result on board

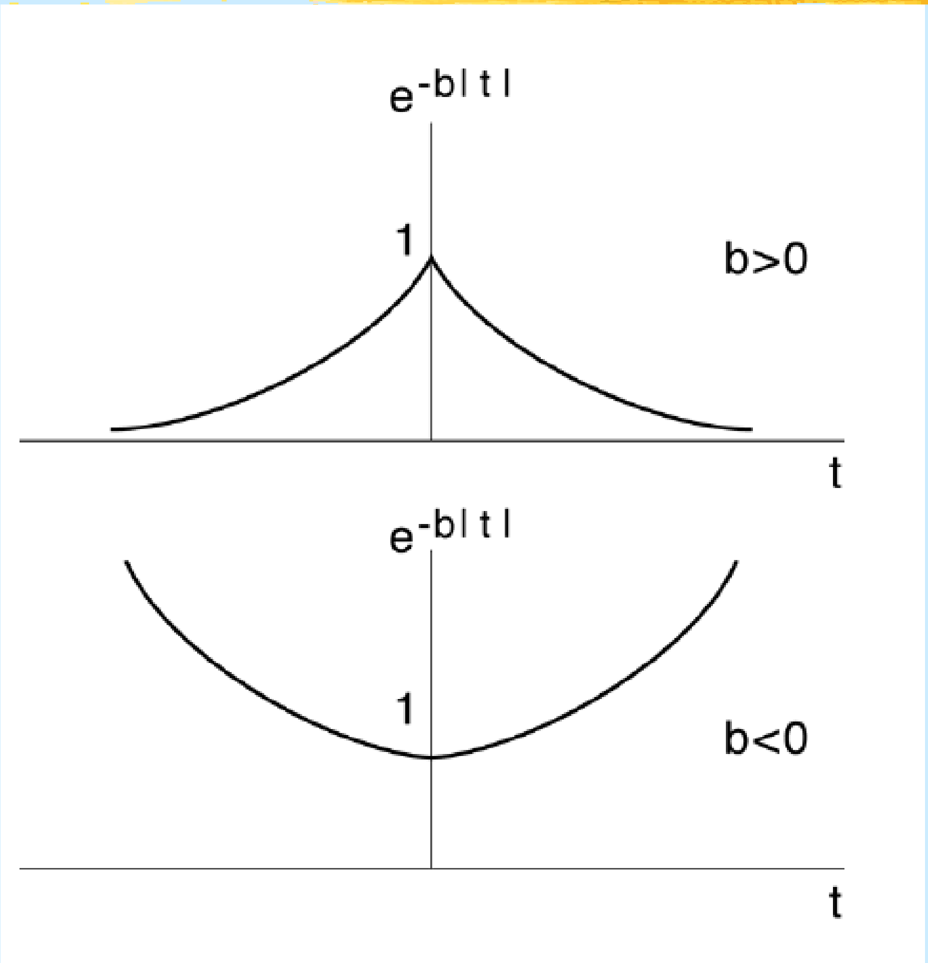
Example #1	Example #2
$X_1(s) = \frac{1}{s+a}, \quad \Re e\{s\} > -\Re e\{a\}$	$X_2(s) = \frac{1}{s+a}, \quad \Re e\{s\} < -\Re e\{a\}$
$x_1(t) = e^{-at}u(t)$ - right-sided signal	$x_2(t) = -e^{-at}u(-t)$ - left-sided signal





Note: same X(s) may correspond to different x(t) depending on ROC!

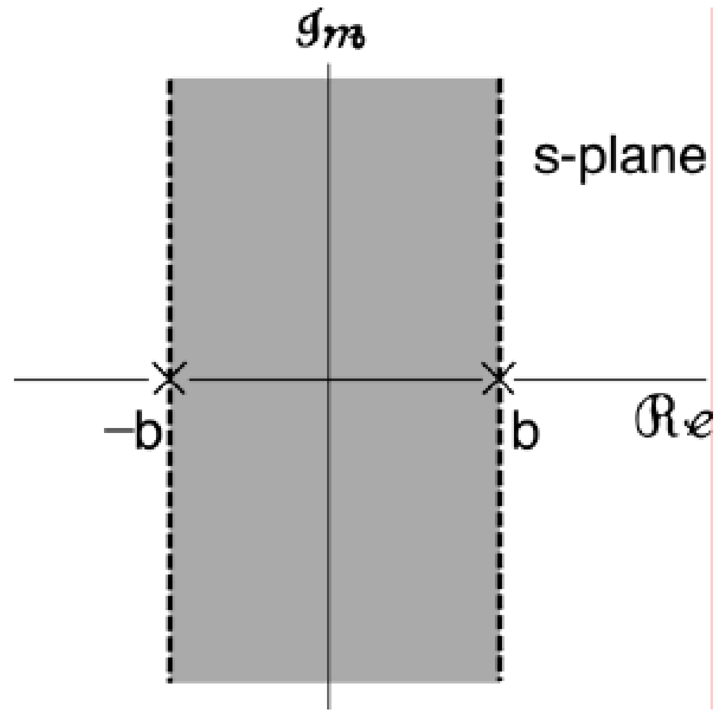
Example 3



$$x(t) = e^{bt}u(-t) + e^{-bt}u(t)$$

$$e^{bt}u(-t) \Downarrow -\frac{1}{s-b}, \ \Re e\{s\} < b \qquad e^{-bt}u(t) \Downarrow \frac{1}{s+b}, \ \Re e\{s\} > -b$$

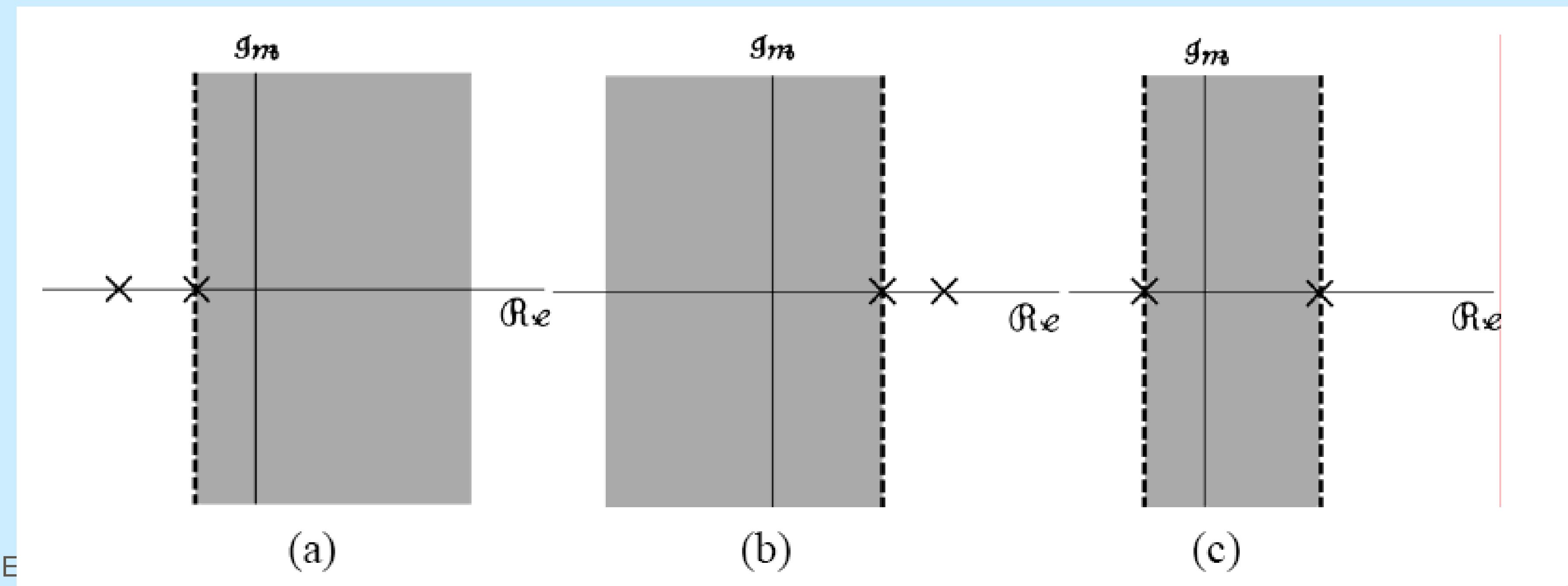
Overlap if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2-b^2}$, with ROC:



What if $b < 0$? $\Rightarrow$ No overlap $\Rightarrow$ No Laplace Transform

General trend of ROC

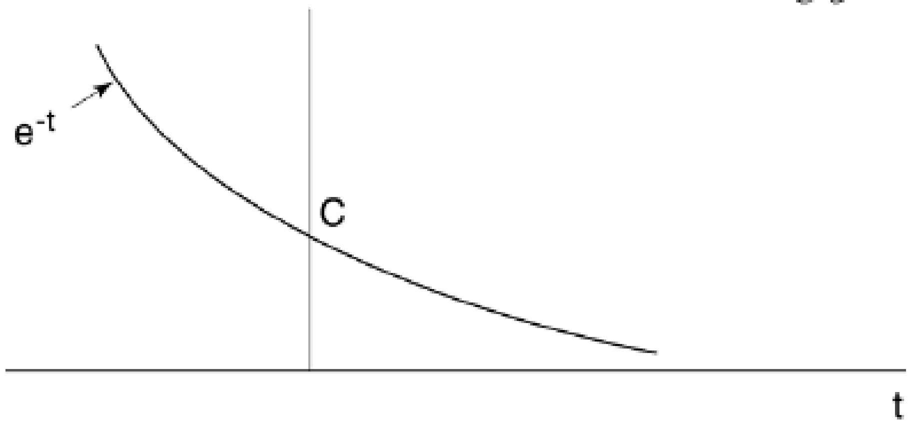
- ROCs are always vertical half planes or stripes, bounded by poles
- Right side signals -> ROC in right half plane
- Left side signals -> ROC in left half plane
- Double sided signals -> ROC in a central stripe, or does not exist



Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all t since $\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \infty$ for all σ

$$|x(t)e^{-\sigma t}| = Ce^{-(\sigma+1)t}$$



(b) $x(t) = e^{j\omega_0 t}$ for all t FT: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty \text{ for all } \sigma$$

$X(s)$ is defined only in ROC; we don't allow impulses in LTs

- Finite duration signals that are absolutely integrable -> ROC contains entire S-plane

Importance of ROC

- X(s) cannot uniquely define x(t)
- Need ROC and X(s)!

Inverse Laplace Transform

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ ROC} \\ &= \mathcal{F}\{x(t)e^{-\sigma t}\} \end{aligned}$$

Fix $\sigma \in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

But $s = \sigma + j\omega$ (σ fixed) $\Rightarrow ds = jd\omega$

$$\Downarrow$$

$j\omega$ in the integral limit should be replaced by $j\infty$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds$$

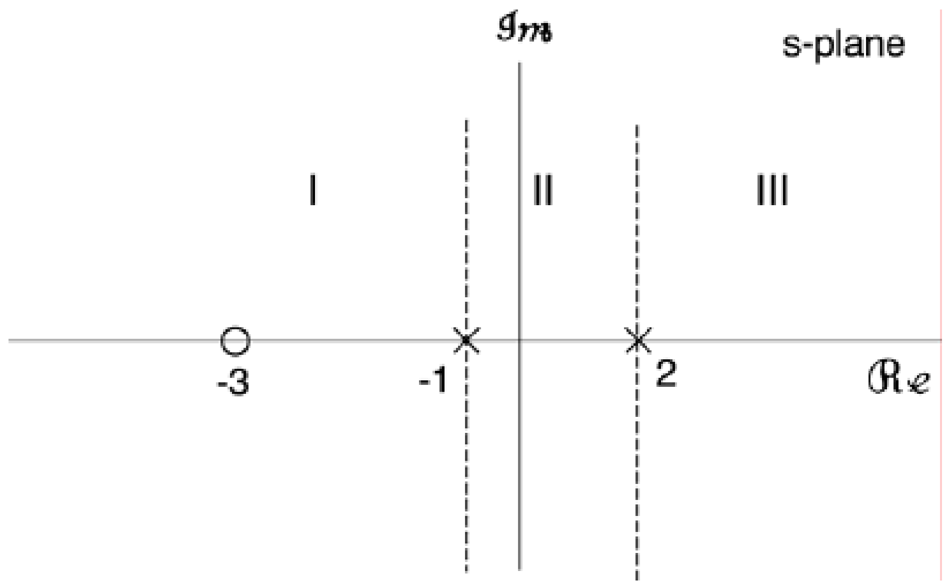
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example:

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3}, \quad B = \frac{5}{3}$$

Three possible ROC's — corresponding to three *different* signals



Recall $\quad \frac{1}{s+a}, \quad \Re e\{s\} < -a \longleftrightarrow -e^{-at}u(-t)$ left-sided

$\frac{1}{s+a}, \quad \Re e\{s\} > -a \longleftrightarrow e^{-at}u(t)$ right-sided

ROC I: — Left-sided signal.

$$x(t) = -Ae^{-t}u(-t) - Be^{2t}u(-t)$$
$$= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \qquad \text{Diverges as } t \to -\infty$$

ROC II: — Two-sided signal, has Fourier Transform.

$$x(t) = Ae^{-t}u(t) - Be^{2t}u(-t)$$
$$= -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \qquad \to 0 \text{ as } t \to \pm\infty$$

ROC III:— Right-sided signal.

$$x(t) = Ae^{-t}u(t) + Be^{2t}u(t)$$
$$= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \qquad \text{Diverges as } t \to +\infty$$