Laplace Transform: Definition and Region of Convergence

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Some slides included are extracted from lecture notes from MIT open courseware http://ocw.mit.edu/OcwWeb/Electrical-Engineering-and-Computer-Science/6-003Fall-2003/CourseHome/
Why do we need another transform?

- Fourier transform cannot handle large (and important) classes of signals and unstable systems, i.e. when

\[ \int_{-\infty}^{\infty} |x(t)| \, dt = \infty \]

- Laplace Transform can be viewed as an extension of the Fourier transform to allow analysis of broader class of signals and systems (including unstable systems!)
Eigen Function of LTI System

- $e^{st}$ is an eigenfunction of any LTI system
  - $s = \sigma + j\omega$ can be complex in general

\[ H(s) = \int_{-\infty}^{\infty} h(t)e^{-st} \, dt \quad \text{(assuming this converges)} \]

- Show on the board
- $H(s)$ is the Laplace transform of $h(t)$!
The (Bilateral) Laplace Transform

\[ x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t)e^{-st} dt = \mathcal{L}\{x(t)\} \]

\[ s = \sigma + j\omega \] is a complex variable
Relation with Fourier Transform

(1) \[ X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt = \mathcal{F}\{x(t)e^{-\sigma t}\} \]

(2) A critical issue in dealing with Laplace transform is convergence:
   - \(X(s)\) generally exists only for some values of \(s\), located in what is called the region of convergence (ROC)

   \[ \text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt < \infty\} \]

(3) If \(s = j\omega\) is in the ROC (i.e. \(\sigma = 0\)), then

   \[ X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\} \]

\[ \text{absolute integrability condition} \]

Depends only on \(\sigma\) not on \(\omega\)
Example 1

\[ x_1(t) = e^{-at}u(t) \]  
(a – an arbitrary real or complex number)

- **Unstable:**
  - no *Fourier Transform*
  - but *Laplace Transform* exists

\[ X_1(s) = \frac{1}{s + a}, \quad \text{Re}\{s\} > -\text{Re}\{a\} \]  

ROC
Derive result on board, sketch ROC for both $a>0$ and $a<0$
Example 2

\[ x_2(t) = -e^{-at}u(-t) \]

Unstable:
- no Fourier Transform
- but Laplace Transform exists

\[ X_2(s) = \frac{1}{s + a}, \quad \text{Re}\{s\} < -\text{Re}\{a\} \]

Same as \( X_1(s) \), but different ROC
• Derive result on board
Example #1

\[ X_1(s) = \frac{1}{s + a}, \quad \Re\{s\} > -\Re\{a\} \]
\[ x_1(t) = e^{-at}u(t) \text{ - right-sided signal} \]

Example #2

\[ X_2(s) = \frac{1}{s + a}, \quad \Re\{s\} < -\Re\{a\} \]
\[ x_2(t) = -e^{-at}u(-t) \text{ - left-sided signal} \]

Note: same \( X(s) \) may correspond to different \( x(t) \) depending on ROC!
Example 3

\[ e^{-bt_1} \]

\[ e^{-bt_1} \]

\[ b > 0 \]

\[ b < 0 \]
\[ x(t) = e^{bt}u(-t) + e^{-bt}u(t) \]

\[ \downarrow \quad \downarrow \]

\[ -\frac{1}{s-b}, \Re\{s\} < b \quad \frac{1}{s+b}, \Re\{s\} > -b \]

Overlap if \( b > 0 \) \( \Rightarrow \) \( X(s) = \frac{-2b}{s^2-b^2} \), with ROC:

What if \( b < 0 \)? \( \Rightarrow \) No overlap \( \Rightarrow \) No Laplace Transform
General trend of ROC

- ROCs are always vertical half planes or stripes, bounded by poles
- Right side signals -> ROC in right half plane
- Left side signals -> ROC in left half plane
- Double sided signals -> ROC in a central stripe, or does not exist
Some signals do not have Laplace Transforms (have no ROC)

(a) \( x(t) = C e^{-t} \) for all \( t \) since 
\[
\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \infty \text{ for all } \sigma
\]

(b) \( x(t) = e^{j\omega_0 t} \) for all \( t \) \( \quad \text{FT: } X(j\omega) = 2\pi \delta(\omega - \omega_0) \)

\[
\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}| dt = \int_{-\infty}^{\infty} e^{-\sigma t} dt = \infty \text{ for all } \sigma
\]

\( X(s) \) is defined only in ROC; we don’t allow impulses in LTs.
 Finite duration signals that are absolutely integrable -> 
ROC contains entire S-plane
Importance of ROC

- $X(s)$ cannot uniquely define $x(t)$
- Need ROC and $X(s)$!
Inverse Laplace Transform

\[ X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ROC} \]

\[ = \mathcal{F}\{x(t)e^{-\sigma t}\} \]

Fix \( \sigma \in \text{ROC} \) and apply the inverse Fourier transform

\[ x(t)e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega \]

\( \downarrow \)

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma+j\omega)t}d\omega \]

But \( s = \sigma + j\omega \) (\( \sigma \) fixed) \( \Rightarrow ds = jd\omega \)

\( \downarrow \)

\[ j\omega \text{ in the integral limit should be replaced by } j\infty \]

\[ x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds \]
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example: $X(s) = \frac{s + 3}{(s + 1)(s - 2)} = \frac{A}{s + 1} + \frac{B}{s - 2}$

$A = -\frac{2}{3}, \quad B = \frac{5}{3}$

Three possible ROC’s — corresponding to three different signals

Recall $\frac{1}{s + a}, \quad \Re\{s\} < -a \leftrightarrow e^{-at}u(-t)$ left-sided

$\frac{1}{s + a}, \quad \Re\{s\} > -a \leftrightarrow e^{-at}u(t)$ right-sided
ROC I: — Left-sided signal.
\[ x(t) = -Ae^{-t}u(-t) - Be^{2t}u(-t) \]
\[ = \left[ \frac{2}{3}e^{-t} - \frac{5}{3}e^{2t} \right] u(-t) \quad \text{Diverges as } t \to -\infty \]

ROC II: — Two-sided signal, has Fourier Transform.
\[ x(t) = Ae^{-t}u(t) - Be^{2t}u(-t) \]
\[ = -\left[ \frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t) \right] \quad \to 0 \text{ as } t \to \pm\infty \]

ROC III: — Right-sided signal.
\[ x(t) = Ae^{-t}u(t) + Be^{2t}u(t) \]
\[ = \left[ -\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t} \right] u(t) \quad \text{Diverges as } t \to +\infty \]