Laplace Transform

- **Review**

- **Slides borrowed from Evans, U. of Texas @ Austin**

Zero-State Response

- **Linear constant coefficient differential equation**

Input $x(t)$ and output $y(t) = y_{zero-input}(t) + y_{zero-state}(t)$

Zero-state response: all initial conditions are zero

$$y(t) \leftrightarrow Y(s) \qquad x(t) \leftrightarrow X(s)$$

$$\frac{d^r}{dt^r} y(t) \leftrightarrow s^r\, Y(s) \qquad \frac{d^k}{dt^k} x(t) \leftrightarrow s^k X(s)$$

Laplace transform both sides of differential equation with all initial conditions being zero and solve for $Y(s)/X(s)$

$$y'(t) + y(t) = x(t) \quad \Longleftrightarrow \quad sY(s) + Y(s) = X(s)$$

$$y(0^-) = 0 \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+1}$$

Transfer Function

- $H(s)$ **is called the transfer function because it describes how input is transferred to the output in a transform domain (*s*-domain in this case)**

$$Y(s) = H(s)\,X(s)$$

$$y(t) = L^{-1}\{H(s)\,X(s)\} = h(t) * x(t) \Rightarrow H(s) = L\{h(t)\}$$

- **Transfer function is Laplace transform of impulse response**

Transfer Function Examples

- **Laplace transform**

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-s\,t}\, dt$$

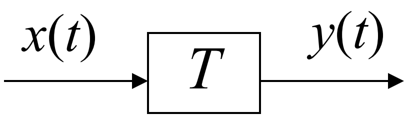
- **Assume input $x(t)$ and output $y(t)$ are causal**
- **Ideal delay of T seconds**

Initial conditions (initial voltages in delay buffer) are zero

$$y(t) = x(t - T)$$

$$Y(s) = X(s)\, e^{-s\,T}$$

$$H(s) = \frac{Y(s)}{X(s)} = e^{-s\,T}$$



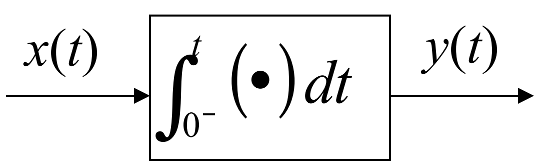
Transfer Function Examples

- **Ideal integrator with** $y(0^-) = 0$

$$y(t) = \int_{0^-}^{t} x(\tau)\,d\tau$$

$$Y(s) = \frac{1}{s}X(s) + \frac{1}{s}y(0^-)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$$

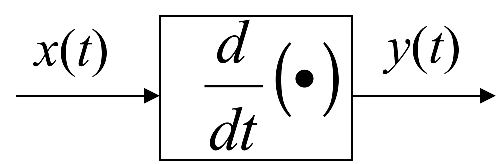


- **Ideal differentiator with** $x(0^-) = 0$

$$y(t) = \frac{d}{dt}x(t)$$

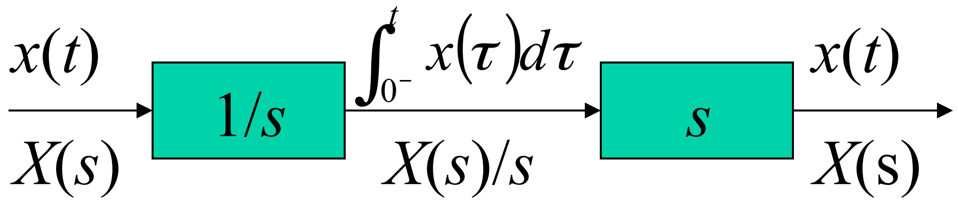
$$Y(s) = s\,X(s) - x(0^-) = s\,X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = s$$

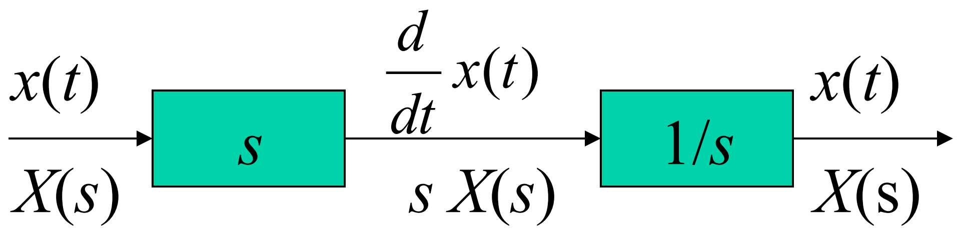


Cascaded Systems

- **Assume input $x(t)$ and output $y(t)$ are causal**
- **Integrator first, then differentiator**



- **Differentiator first, then integrator**



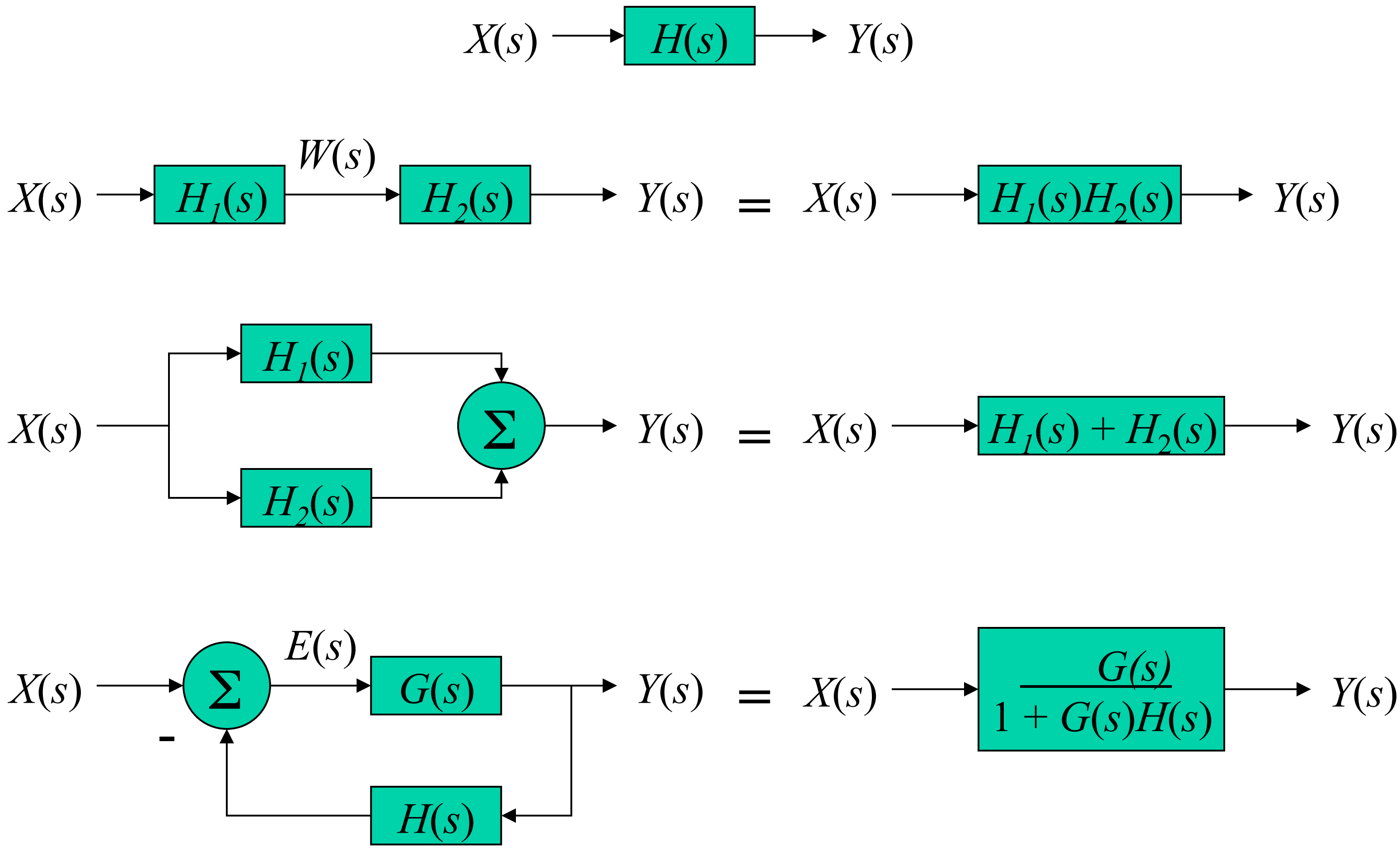
- **Common transfer functions**

A constant (finite impulse response)

A polynomial (finite impulse response)

Ratio of two polynomials (infinite impulse response)

Block Diagrams

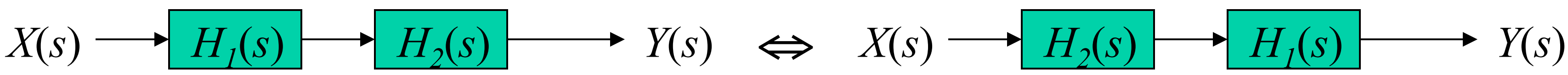


Cascade and Parallel Connections

- **Cascade**

$W(s) = H_1(s)\,X(s)$ $\qquad\qquad$ $Y(s) = H_2(s)W(s)$

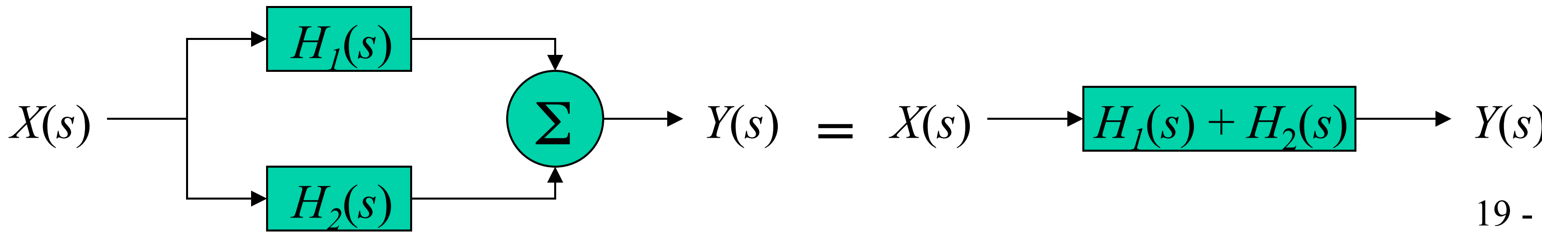
$Y(s) = H_1(s)\,H_2(s)\,X(s) \Rightarrow Y(s)/X(s) = H_1(s)H_2(s)$



One can switch the order of the cascade of two LTI systems if both LTI systems compute to exact precision

- **Parallel Combination**

$Y(s) = H_1(s)X(s) + H_2(s)X(s) \rightarrow Y(s)/X(s) = H_1(s) + H_2(s)$



Feedback Connection

- **Governing equations**

$$E(s) = F(s) - H(s)Y(s)$$

$$Y(s) = G(s)E(s)$$

- **Combining equations**

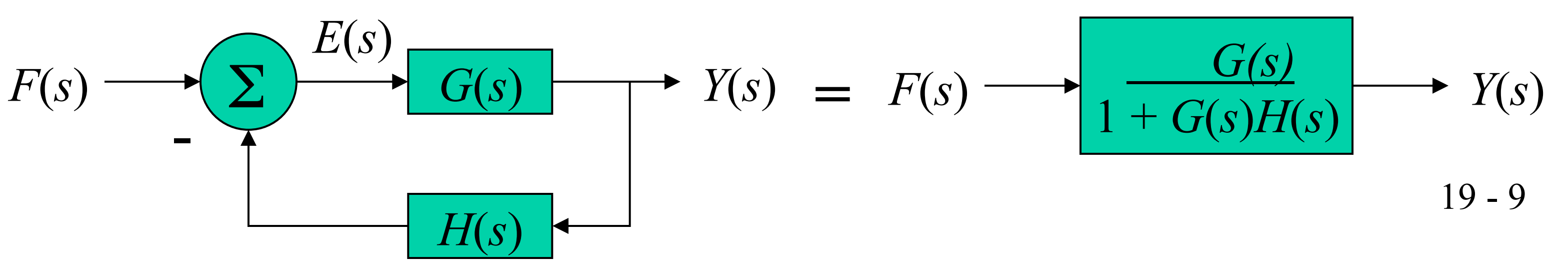
$$Y(s) = G(s)\left[\, F(s) - H(s)Y(s) \,\right]$$

$$Y(s) + G(s)H(s)Y(s) = G(s)F(s)$$

$$Y(s) = \frac{G(s)}{1 + G(s)H(s)} F(s)$$

- **What happens if $H(s)$ is a constant K?**

Choice of K controls all poles in transfer function



External Stability Conditions

- **Bounded-input bounded-output stability**

  Zero-state response given by $h(t) * x(t)$

  Two choices: BIBO stable or BIBO unstable

- **Remove common factors in transfer function $H(s)$**
- **If all poles of $H(s)$ in left-hand plane,**

  All terms in $h(t)$ are decaying exponentials

  $h(t)$ is absolutely integrable and system is BIBO stable

- **Example: BIBO stable but asymptotically unstable**

$$H(s) = \left(\frac{s-1}{s^2-1}\right) = \left(\frac{1}{s-1}\right)\left(\frac{s-1}{s+1}\right) = \left(\frac{1}{s+1}\right)$$

Based on slide by Prof. Adnan Kavak

Internal Stability Conditions

- **Stability based on zero-input solution**
- **Asymptotically stable if and only if**

  Characteristic roots are in left-hand plane (LHP)

  Roots may be repeated or non-repeated
- **Unstable if and only if**

  (i) at least characteristic root in right-hand plane and/or

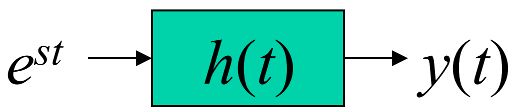
  (ii) repeated characteristic roots are on imaginary axis
- **Marginally stable if and only if**

  There are no characteristic roots in right-hand plane and

  Some non-repeated roots are on imaginary axis

Based on slide by Prof. Adnan Kavak

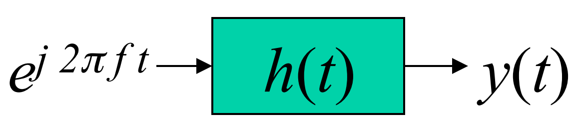
Frequency-Domain Interpretation



- $y(t) = H(s)\, e^{s\,t}$
  for a particular value of *s*

$$y(t) = h(t) * e^{s\,t}$$
$$= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau$$
$$= e^{s\,t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\,\tau} d\tau}_{H(s)}$$

- **Recall definition of frequency response:**



$$y(t) = h(t) * e^{j\,2\pi\,f\,t}$$
$$= \int_{-\infty}^{\infty} h(\tau) e^{j\,2\pi\,f\,(t-\tau)} d\tau$$
$$= e^{j\,2\pi\,f\,t} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{j\,2\pi\,f\,\tau} d\tau}_{H(f)}$$

Frequency-Domain Interpretation

- **Generalized frequency:** $s = \sigma + j\,2\,\pi\,f$
- **We may convert transfer function into frequency response by if and only if region of convergence of** $H(s)$ **includes the imaginary axis**

$$H_{\text{freq}}(f) = H(s)\Big|_{s=j2\pi f}$$

- **What about** $h(t) = u(t)$**?** $H(s) = \frac{1}{s}$ for $\text{Re}\{s\} > 0$

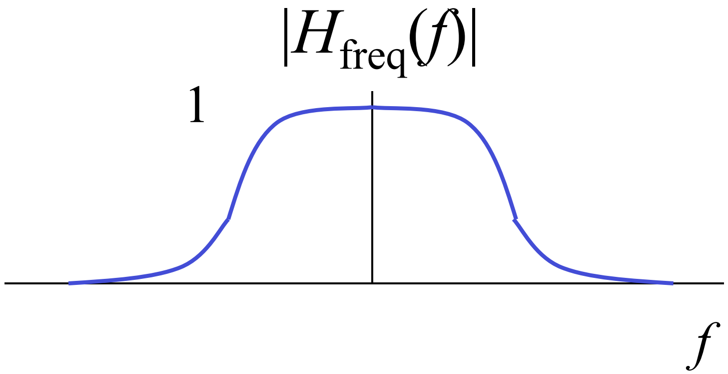
  We *cannot* convert $H(s)$ to a frequency response

  However, this system has a frequency response

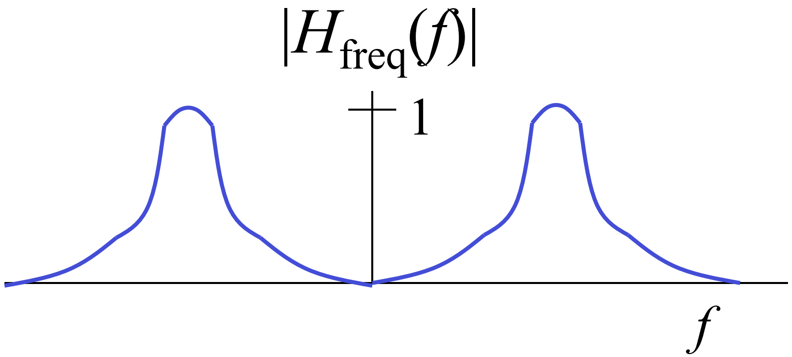
- **What about** $h(t) = \delta(t)$**?** $H(s) = 1$ for all $s \Rightarrow H_{\text{freq}}(f) = 1$

Frequency Selectivity in Filters

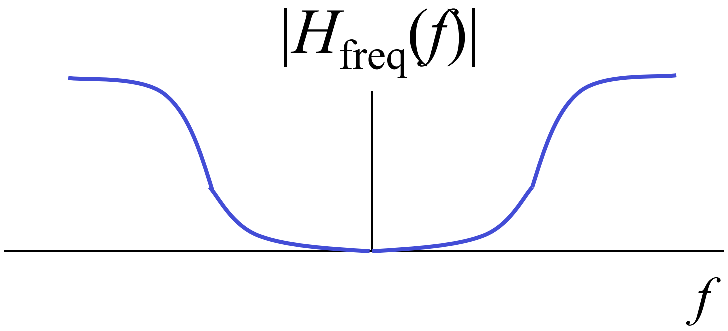
- **Lowpass filter**



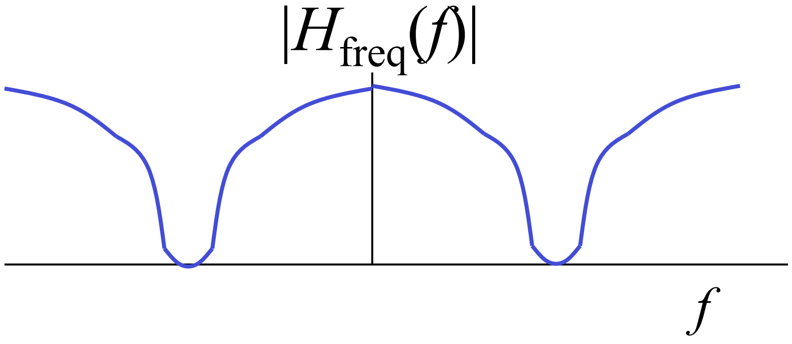
- **Bandpass filter**



- **Highpass filter**



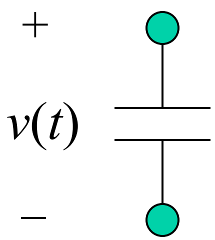
- **Bandstop filter**



Linear time-invariant *filters* are BIBO stable

Passive Circuit Elements

- **Laplace transforms with zero-valued initial conditions**
- **Capacitor**



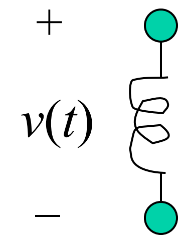
$$i(t) = C \frac{dv}{dt}$$

$$I(s) = C\, s\, V(s)$$

$$V(s) = \frac{1}{C\, s} I(s)$$

$$H(s) = \frac{V(s)}{I(s)} = \frac{1}{C\, s}$$

- **Inductor**

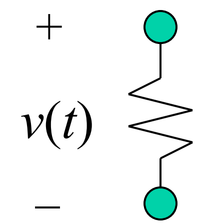


$$v(t) = L \frac{di}{dt}$$

$$V(s) = L\, s\, I(s)$$

$$H(s) = \frac{V(s)}{I(s)} = L\, s$$

- **Resistor**

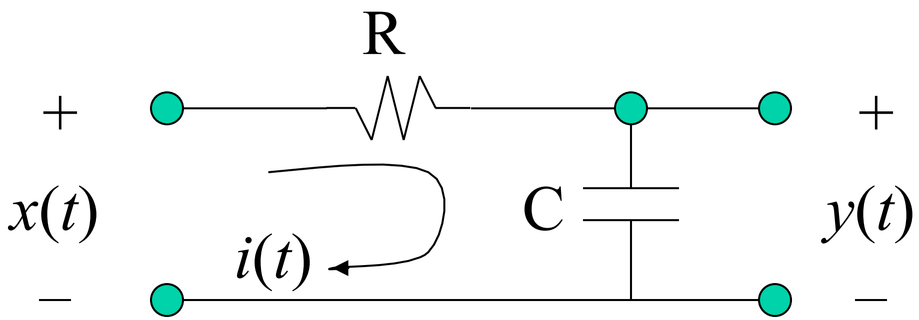


$$v(t) = R\, i(t)$$

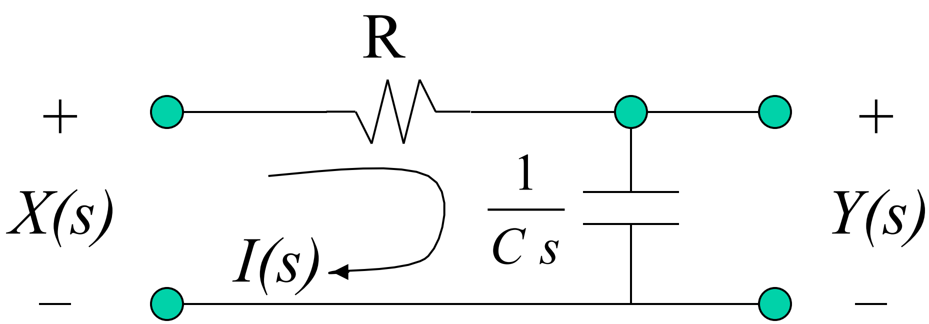
$$V(s) = R\, I(s)$$

$$H(s) = \frac{V(s)}{I(s)} = R$$

First-Order RC Lowpass Filter



Time domain



Laplace domain

$$I(s) = \frac{X(s)}{R + \frac{1}{C\,s}}$$

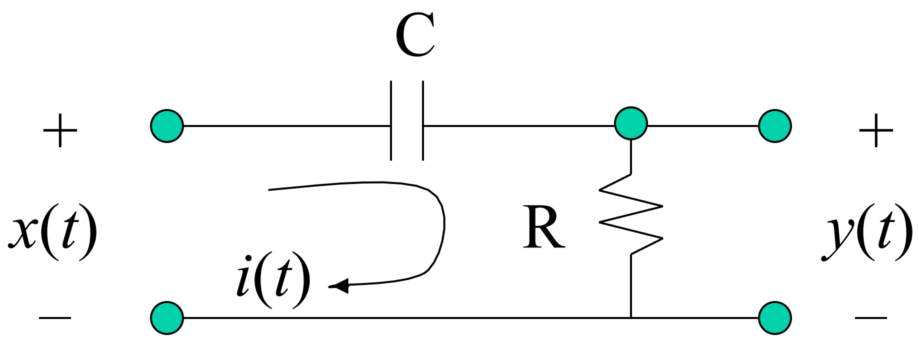
$$Y(s) = \frac{1}{C\,s} I(s)$$

$$Y(s) = \frac{\frac{1}{C\,s}}{R + \frac{1}{C\,s}} X(s)$$

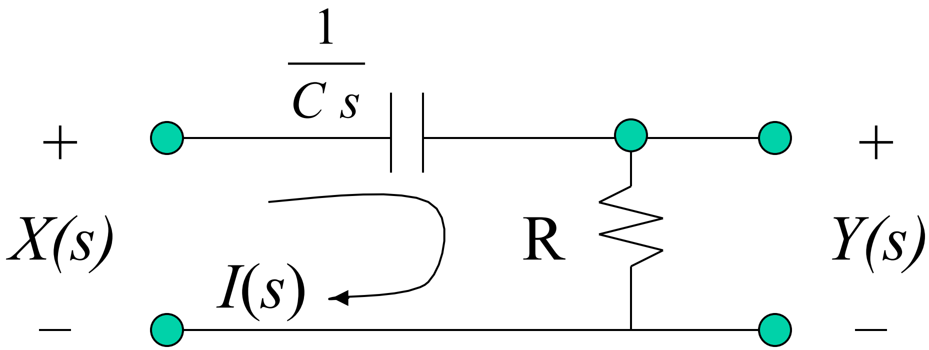
$$\frac{Y(s)}{X(s)} = \frac{\frac{1}{R\,C}}{s + \frac{1}{R\,C}}$$

Plot Bode plot – Amplitude & Phase?

First-Order RC Highpass Filter



Time domain



Laplace domain

Plot Bode plot – Amplitude & Phase?

$$I(s) = \frac{X(s)}{R + \frac{1}{C\,s}}$$

$$Y(s) = R\,I(s)$$

$$Y(s) = \frac{R}{R + \frac{1}{C\,s}} X(s)$$

$$\frac{Y(s)}{X(s)} = \frac{s}{s + \frac{1}{R\,C}}$$

Frequency response is also an example of a notch filter

Passive Circuit Elements

- **Laplace transforms with non-zero initial conditions**
- **Capacitor**

$$i(t) = C\,\frac{dv}{dt}$$

$$I(s) = C\left[s\,V(s) - v(0^-)\right]$$

$$V(s) = \frac{1}{C\,s}\,I(s) + \frac{v(0^-)}{s}$$

$$= \frac{1}{C\,s}\left[I(s) + C\,v(0^-)\right]$$

- **Inductor**

$$V(t) = L\,\frac{di}{dt}$$

$$V(s) = L\left[s\,I(s) - i(0^-)\right]$$

$$= L\,s\,I(s) - L\,i(0^-)$$

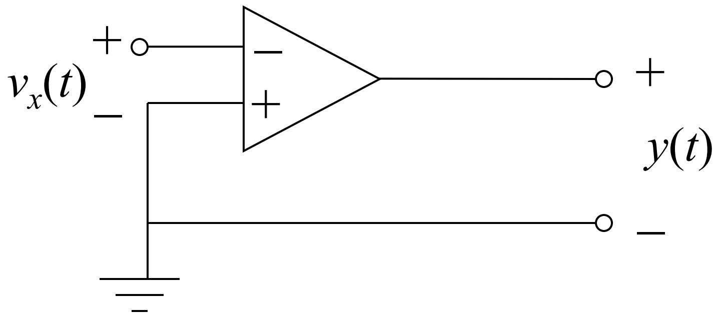
$$= L\,s\left[I(s) - \frac{i(0^-)}{s}\right]$$

Operational Amplifier

- **Ideal case: model this nonlinear circuit as linear and time-invariant**

  Input impedance is extremely high (considered infinite)

  $v_x(t)$ is very small (considered zero)



Operational Amplifier Circuit

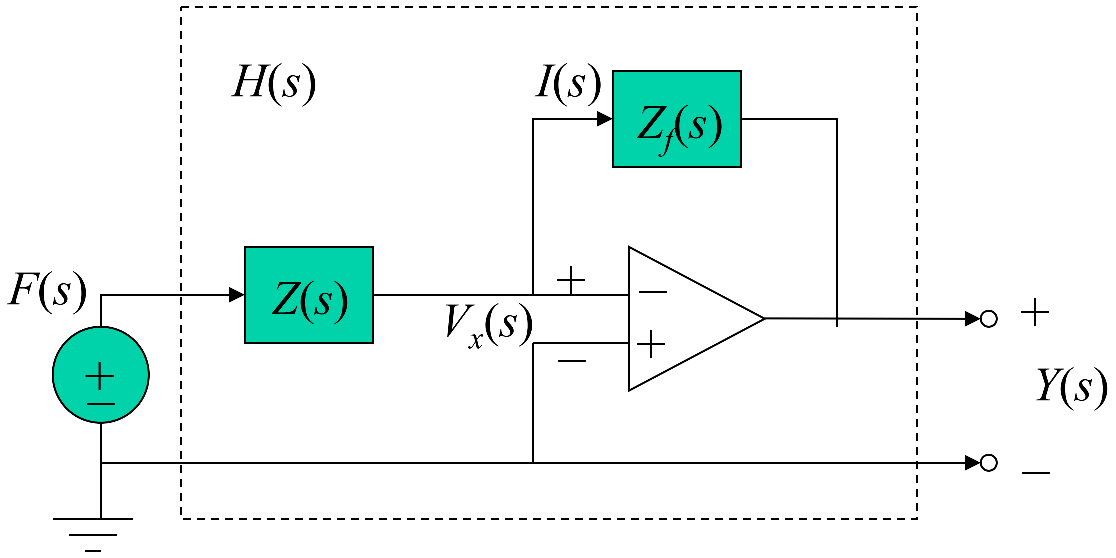
- **Assuming that** $V_x(\text{s}) = 0$**,**

$$Y(s) = -I(s)Z_f(s)$$

$$I(s) = \frac{F(s)}{Z(s)}$$

$$Y(s) = -\frac{Z_f(s)}{Z(s)}F(s)$$

$$H(s) = \frac{Y(s)}{F(s)} = -\frac{Z_f(s)}{Z(s)}$$



- **How to realize a gain of –1?**
- **How to realize a gain of 10?**

Differentiator

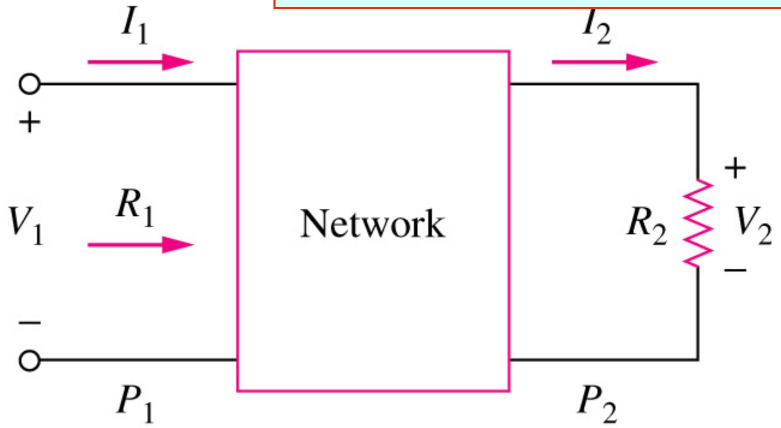
- **A differentiator amplifies high frequencies, e.g. high-frequency components of noise:**

  $H(s) = s$ for all values of s (see next slide)

  Frequency response is $H(f) = j\, 2\, \pi\, f \;\Rightarrow |\, H(f)\, | = 2\, \pi\, |\, f\, |$
- **Noise has equal amounts of low and high frequencies up to a physical limit**
- **A differentiator may amplify noise to drown out a signal of interest**
- **In analog circuit design, one would generally use integrators instead of differentiators**

DECIBEL SCALE

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1}$$ is defined as Decibel Gain



$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} = 20\log_{10}\frac{V_2}{V_1}$$

$$G_{dB} = 10\log_{10}\frac{P_2}{P_1} = 10\log_{10}\frac{V_2^{\,2}/R_2}{V_1^{\,2}/R_1} = 10\log_{10}\left(\frac{V_2}{V_1}\right)^2 + 10\log_{10}\frac{R_1}{R_2}$$

$$= 20\log_{10}\frac{V_2}{V_1} - 10\log_{10}\frac{R_2}{R_1} = 20\log_{10}\frac{V_2}{V_1} \qquad \text{if} \quad R_2 = R_1$$

DECIBEL SCALE

- The DECIBEL value is a logarithmic measurement of the ratio of one variable to another of the same type.
- Decibel value has no dimension.
- It is used for voltage, current and power gains.

Magnitude H	Decibel Value H_{dB}
0.001	-60
0.01	-40
0.1	-20
0.5	-6
$1/\sqrt{2}$	-3
1	0
$\sqrt{2}$	3
2	6
10	20
20	26
100	40

$$H_{dB} = 20\log_{10} H = 20\log_{10}\frac{V_2}{V_1}$$

Some Properties of Logarithms

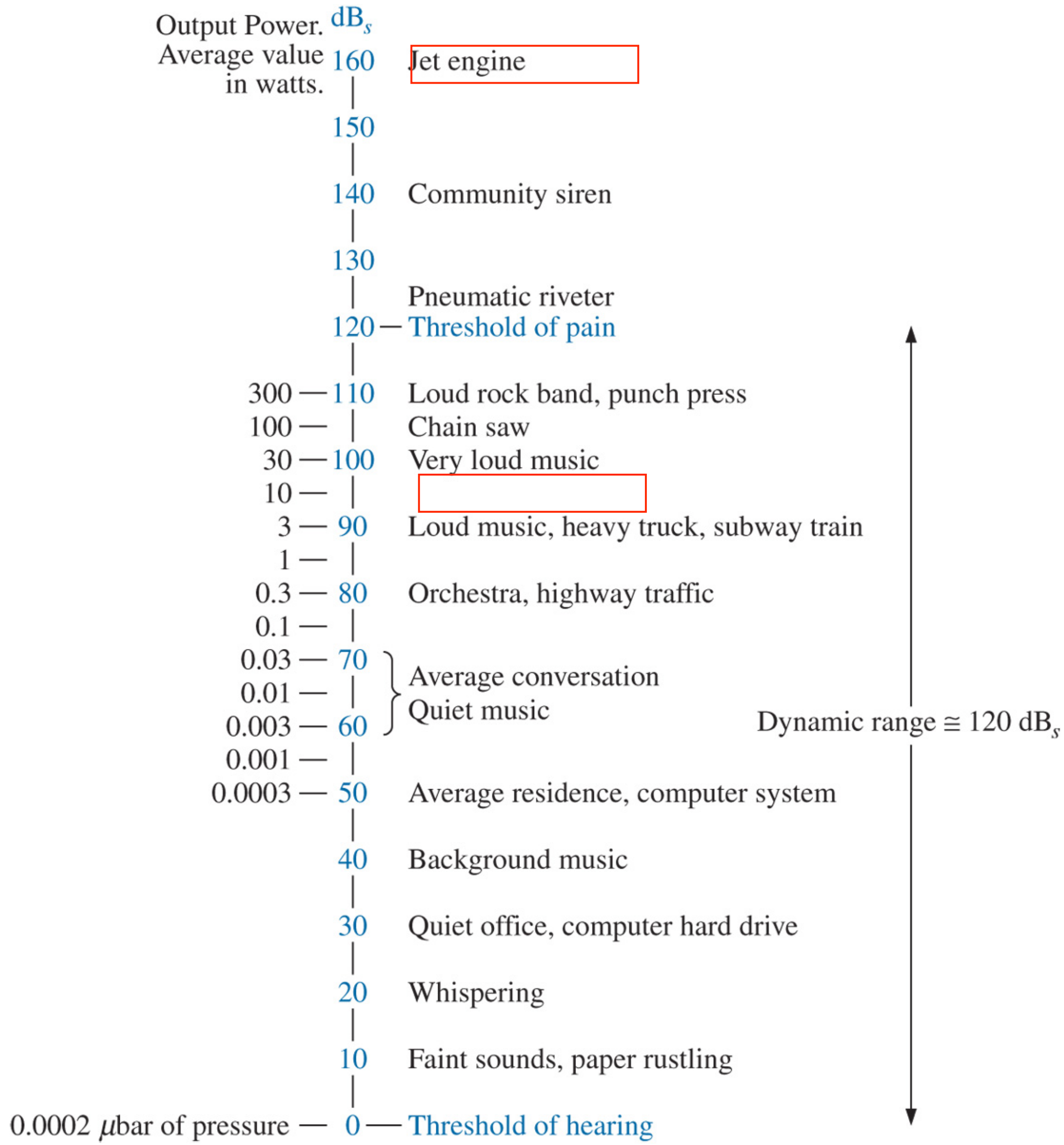
$$\log P_1 P_2 = \log P_1 + \log P_2$$

$$\log\left(\frac{P_1}{P_2}\right) = \log P_1 - \log P_2$$

$$\log P^n = n\log P$$

$$\log 1 = 0$$

Typical Sound Levels and Their Decibel Levels.



EXAMPLE 1 Construct Bode plots for

$$H(\omega) = \frac{200\, j\omega}{(j\omega + 2)(j\omega + 10)}$$

- Express transfer function in Standard form.

STANDARD FORM $H(\omega) = \frac{10\, j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$

- Express the magnitude and phase responses.

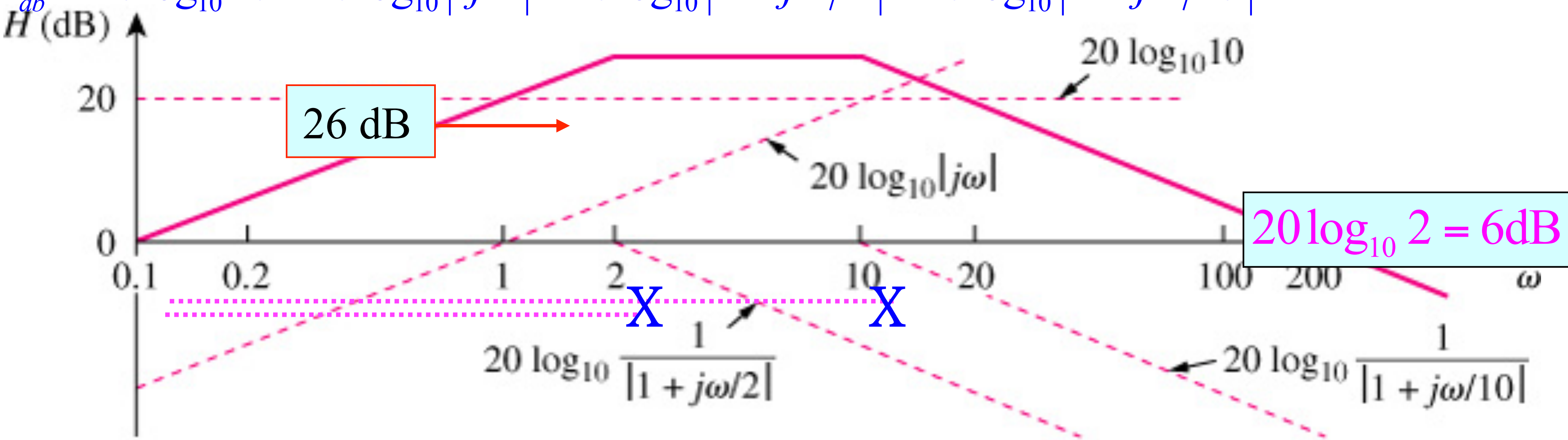
$$H_{db} = 20\log_{10} 10 + 20\log_{10}|\,j\omega\,| - 20\log_{10}|1 + j\omega/2| - 20\log_{10}|1 + j\omega/10|$$

$$\phi = 90^\circ - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/10)$$

- Two corner frequencies at ω=2, 10 and a zero at the origin ω=0.
- Sketch each term and add to find the total response.

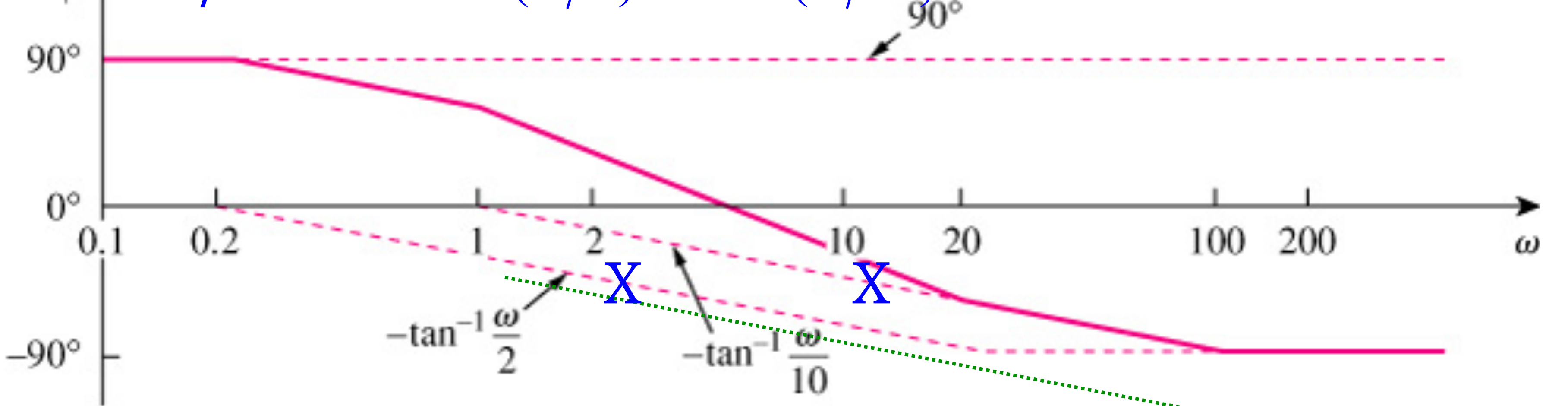
EXAMPLE 1 Construct Bode plots for $H(\omega) = \frac{200 j\omega}{(j\omega+2)(j\omega+10)}$

$$H_{db} = 20\log_{10} 10 + 20\log_{10}| j\omega| - 20\log_{10}|1 + j\omega/2| - 20\log_{10}|1 + j\omega/10|$$



(a)

$$\phi = 90° - \tan^{-1}(\omega/2) - \tan^{-1}(\omega/10)$$

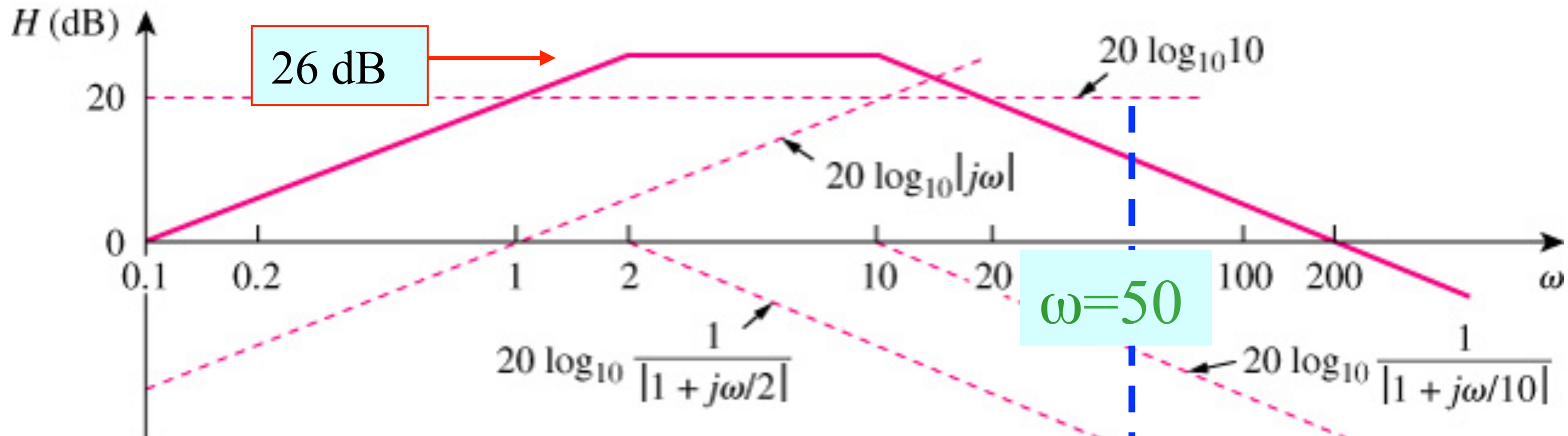


(b)

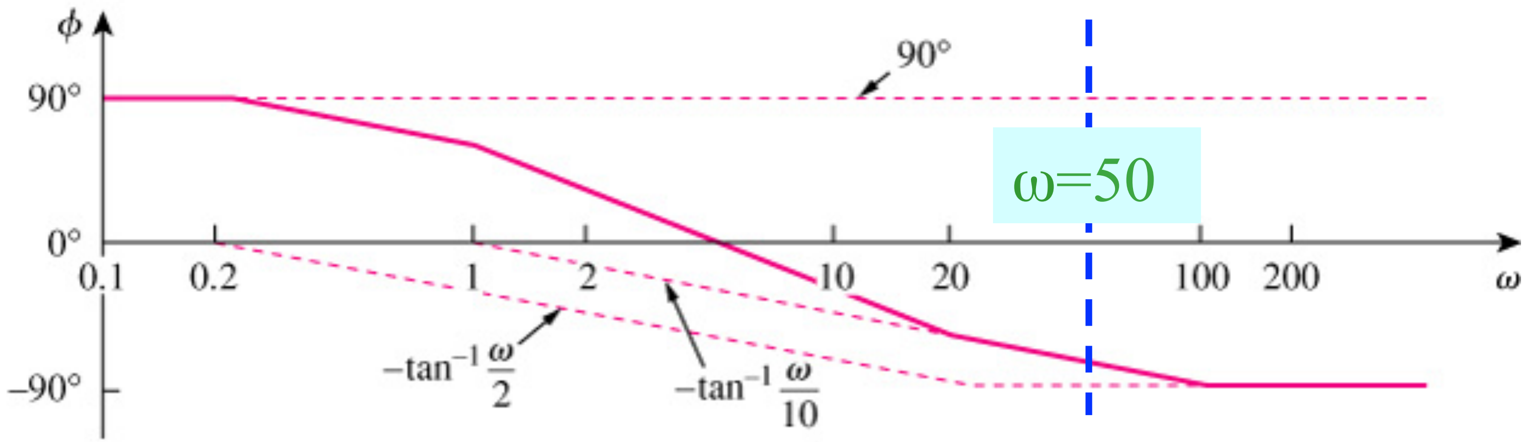
EXAMPLE 2 Continued: Let us calculate $|H|$ and ϕ at ω=50 rad/sec graphically.

$$H(50) = H(10) - 20\log_{10}(50/10) = 26 - 20\times 0.7 = 26 - 14 = 12\,dB$$

$$\phi(50) = 90° - 45°\times\log_{10}(1/0.2) - 90°\times\log_{10}(20/1) - 45°\times\log_{10}(50/20)$$
$$= 90° - 45°\times 0.7 - 90°\times 1.3 - 45°\times 0.4 = 90° - 31.5° - 117° - 18° = -76.5°$$



(a)



(b)

EXAMPLE 1 Construct Bode plots for $H(\omega) = \dfrac{j\omega + 10}{j\omega\left(j\omega + 5\right)^2}$

- Express transfer function in Standard form.

STANDARD FORM

$$H(\omega) = \frac{0.4(1 + j\omega/10)}{j\omega(1 + j\omega/5)^2}$$

- Express the magnitude and phase responses.

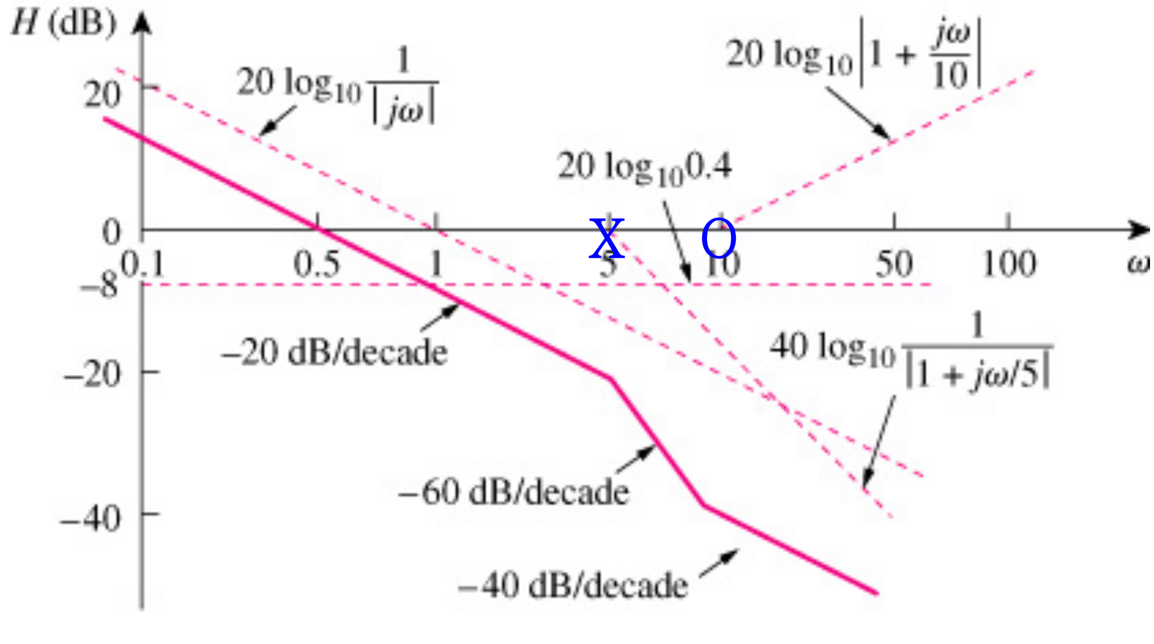
$$H_{db} = 20\log_{10} 0.4 + 20\log_{10}\left|1 + j\omega/10\right| - 20\log_{10}\left|j\omega\right| - 40\log_{10}\left|1 + j\omega/5\right|$$

$$\phi = 0° + \tan^{-1}(\omega/10) - 90° - 2\tan^{-1}(\omega/5)$$

- Two corner frequencies at ω=5, 10 and a zero at ω=10.
- The pole at ω=5 is a double pole. The slope of the magnitude is -40 dB/decade and phase has slope -90 degree/decade.
- Sketch each term and add to find the total response.

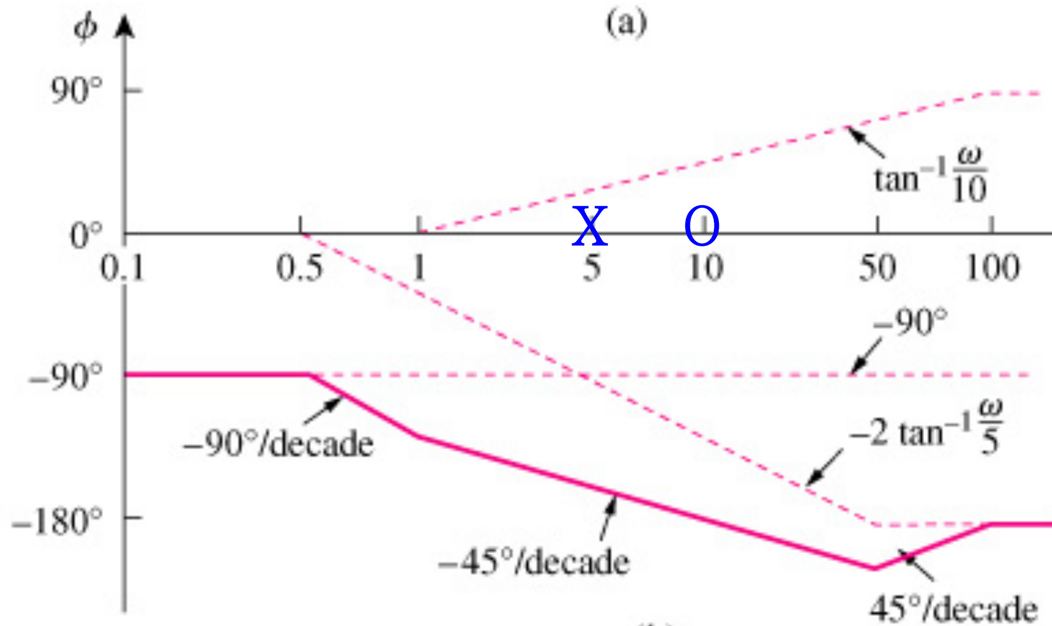
EXAMPLE 3 Construct Bode plots for $H(\omega)=\dfrac{j\omega+10}{j\omega(j\omega+5)^2}$

$$H_{db} = 20\log_{10} 0.4 + 20\log_{10}|1+j\omega/10| - 20\log_{10}|j\omega| - 40\log_{10}|1+j\omega/5|$$



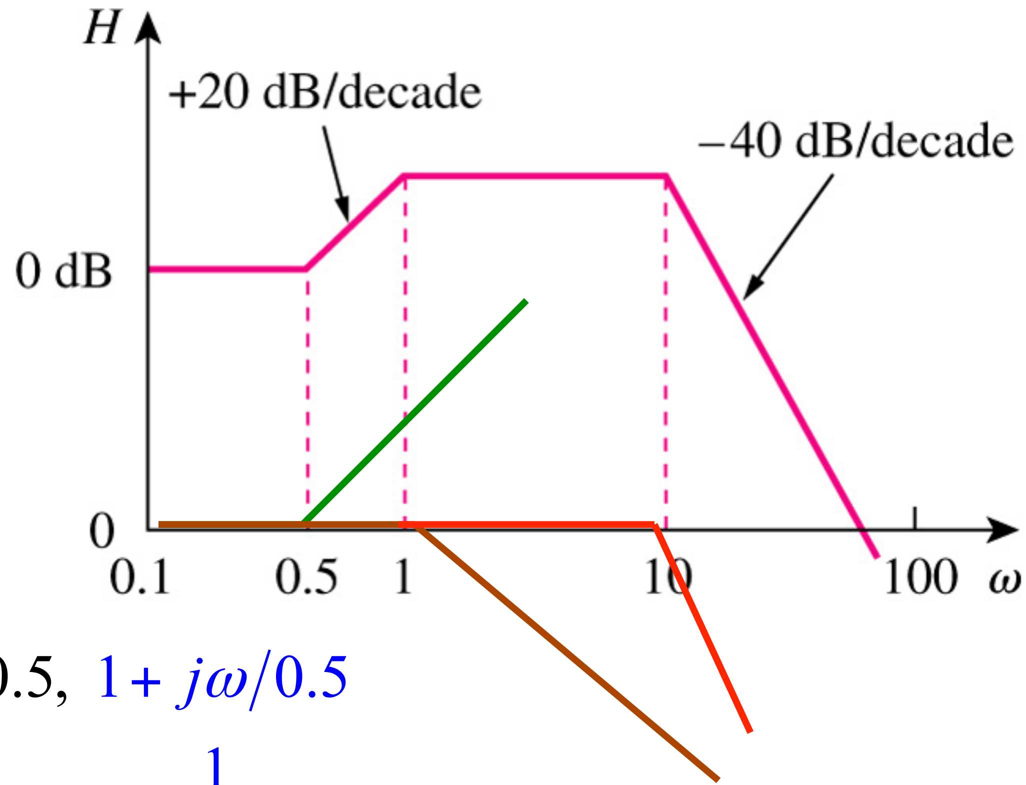
(a)

$$\phi = 0° + \tan^{-1}(\omega/10) - 90° - 2\tan^{-1}(\omega/5)$$



(b)

PRACTICE PROBLEM 4 Obtain the transfer function for the Bode plot given.



A zero at $\omega = 0.5$, $1 + j\omega/0.5$

A pole at $\omega = 1$, $\dfrac{1}{1 + j\omega/1}$

Two poles at $\omega = 10$, $\dfrac{1}{(1 + j\omega/10)^2}$

$$H(\omega) = \frac{1 + j\omega/0.5}{(1 + j\omega/1)(1 + j\omega/10)^2} = \frac{(1/0.5)(0.5 + j\omega)}{(1/100)(1 + j\omega)(10 + j\omega)^2} = \frac{200(s + 0.5)}{(s + 1)(s + 10)^2}$$