II

Physical System Modeling

7

Modeling Electromechanical Systems

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7.1 Introduction

Mechatronics describes the integration of mechanical, electromagnetic, and computer elements to produce devices and systems that monitor and control machine and structural systems. Examples include familiar consumer machines such as VCRs, automatic cameras, automobile air bags, and cruise control devices. A distinguishing feature of modern mechatronic devices compared to earlier controlled machines is the miniaturization of electronic information processing equipment. Increasingly computer and electronic sensors and actuators can be embedded in the structures and machines. This has led to the need for integration of mechanical and electrical design. This is true not only for sensing and signal processing but also for actuator design. In human size devices, more powerful magnetic materials and superconductors have led to the replacement of hydraulic and pneumatic actuators with servo motors, linear motors, and other electromagnetic actuators. At the material scale and in microelectromechanical systems (MEMS), electric charge force actuators, piezoelectric actuators, and ferroelectric actuators have made great strides.

While the materials used in electromechanical design are often new, the basic dynamic principles of Newton and Maxwell still apply. In spatially extended systems one must solve continuum problems using the theory of elasticity and the partial differential equations of electromagnetic field theory. For many applications, however, it is sufficient to use lumped parameter modeling based on i) rigid body dynamics for inertial components, ii) Kirchhoff circuit laws for current-charge components, and iii) magnet circuit laws for magnetic flux devices.

In this chapter we will examine the basic modeling assumptions for inertial, electric, and magnetic circuits, which are typical of mechatronic systems, and will summarize the dynamic principles and interactions between the mechanical motion, circuit, and magnetic state variables. We will also illustrate these principles with a few examples as well as provide some bibliography to more advanced references in electromechanics.

7.2 Models for Electromechanical Systems

The fundamental equations of motion for physical continua are partial differential equations (PDEs), which describe dynamic behavior in both time and space. For example, the motions of strings, elastic beams and plates, fluid flow around and through bodies, as well as magnetic and electric fields require both spatial and temporal information. These equations include those of elasticity, elastodynamics, the Navier–Stokes equations of fluid mechanics, and the Maxwell–Faraday equations of electromagnetics. Electromagnetic field problems may be found in Jackson (1968). Coupled field problems in electric fields and fluids may be found in Melcher (1980) and problems in magnetic fields and elastic structures may be found in the monograph by Moon (1984). This short article will only treat solid systems.

Many practical electromechanical devices can be modeled by lumped physical elements such as mass or inductance. The equations of motion are then integral forms of the basic PDEs and result in coupled ordinary differential equations (ODEs). This methodology will be explored in this chapter. Where physical problems have spatial distributions, one can often separate the problem into spatial and temporal parts called *separation of variables*. The spatial description is represented by a finite number of spatial or eigenmodes each of which has its modal amplitude. This method again results in a set of ODEs. Often these coupled equations can be understood in the context of simple lumped mechanical masses and electric and magnetic circuits.

7.3 Rigid Body Models

Kinematics of Rigid Bodies

Kinematics is the description of motion in terms of position vectors $\mathbf{r}$, velocities $\mathbf{v}$, acceleration $\mathbf{a}$, rotation rate vector ω, and generalized coordinates $\{q_k(t)\}$ such as relative angular positions of one part to another in a machine (Fig. 7.1). In a rigid body one generally specifies the position vector of one point, such as the center of mass $\mathbf{r}_c$, and the velocity of that point, say $\mathbf{v}_c$. The angular position of a rigid body is specified by angle sets call Euler angles. For example, in vehicles there are pitch, roll, and yaw angles (see, e.g., Moon, 1999). The angular velocity vector of a rigid body is denoted by ω. The velocity of a point in a rigid body other than the center of mass, $\mathbf{r}_p = \mathbf{r}_c + \rho$, is given by

$$\mathbf{v}_P = \mathbf{v}_c + \omega \times \rho \tag{7.1}$$

where the second term is a vector cross product. The angular velocity vector ω is a property of the entire rigid body. In general a rigid body, such as a satellite, has six degrees of freedom. But when machine elements are modeled as a rigid body, kinematic constraints often limit the number of degrees of freedom.

Constraints and Generalized Coordinates

Machines are often collections of rigid body elements in which each component is constrained to have one degree of freedom relative to each of its neighbors. For example, in a multi-link robot arm shown in Fig. 7.2, each rigid link has a revolute degree of freedom. The degrees of freedom of each rigid link are constrained by bearings, guides, and gearing to have one type of relative motion. Thus, it is convenient

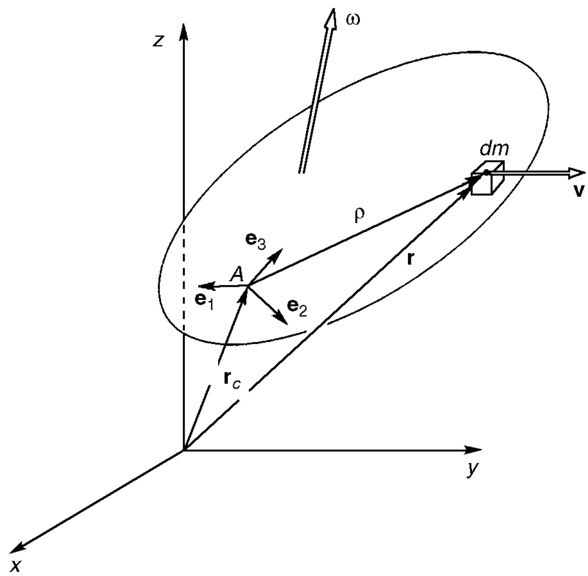

FIGURE 7.1 Sketch of a rigid body with position vector, velocity, and angular velocity vectors.

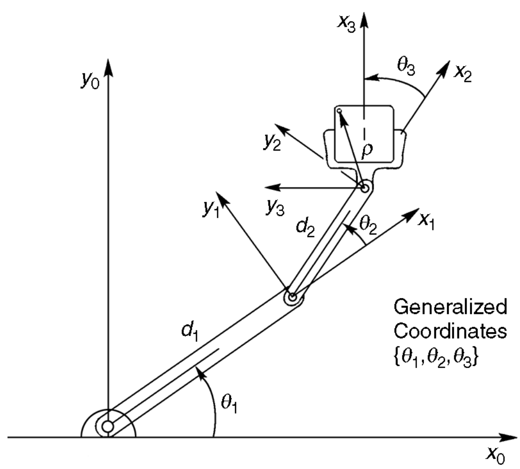

FIGURE 7.2 Multiple link robot manipulator arm.

to use these generalized motions $\{q_k: k = 1,\ldots,K\}$ to describe the dynamics. It is sometimes useful to define a vector or matrix, $\mathbf{J}(q_k)$, called a *Jacobian*, that relates velocities of physical points in the machine to the generalized velocities $\{\dot{q}_k\}$. If the position vector to some point in the machine is $\mathbf{r}_P(q_k)$ and is determined by geometric constraints indicated by the functional dependence on the $\{q_k(t)\}$, then the velocity of that point is given by

$$\mathbf{v}_P = \sum \frac{\partial \mathbf{r}_P}{\partial q_r}\dot{q}_r = \mathbf{J} \cdot \dot{\mathbf{q}} \tag{7.2}$$

where the sum is on the number of generalized degrees of freedom K. The three-by-K matrix $\mathbf{J}$ is called a *Jacobian* and $\dot{\mathbf{q}}$ is a $K \times 1$ vector of generalized coordinates. This expression can be used to calculate

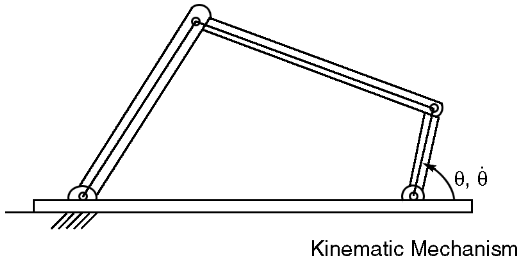

FIGURE 7.3 Example of a kinematic mechanism.

the kinetic energy of the constrained machine elements, and using Lagrange's equations discussed below, derive the equations of motion (see also Moon, 1999).

Kinematic versus Dynamic Problems

Some machines are constructed in a closed kinematic chain so that the motion of one link determines the motion of the rest of the rigid bodies in the chain, as in the four-bar linkage shown in Fig. 7.3. In these problems the designer does not have to solve differential equations of motion. Newton's laws are used to determine forces in the machine, but the motions are *kinematic*, determined through the geometric constraints.

In open link problems, such as robotic devices (Fig. 7.2), the motion of one link does not determine the dynamics of the rest. The motions of these devices are inherently *dynamic*. The engineer must use both the kinematic constraints (7.2) as well as the Newton–Euler differential equation of motion or equivalent forms such as Lagrange's equation discussed below.

7.4 Basic Equations of Dynamics of Rigid Bodies

In this section we review the equations of motion for the mechanical plant in a mechatronics system. This plant could be a system of rigid bodies such as in a serial robot manipulator arm (Fig. 7.2) or a magnetically levitated vehicle (Fig. 7.4), or flexible structures in a MEMS accelerometer. The dynamics of flexible structural systems are described by PDEs of motion. The equation for rigid bodies involves Newton's law for the motion of the center of mass and Euler's extension of Newton's laws to the angular momentum of the rigid body. These equations can be formulated in many ways (see Moon, 1999):

1. Newton–Euler equation (vector method)
2. Lagrange's equation (scalar-energy method)
3. D'Alembert's principle (virtual work method)
4. Virtual power principle (Kane's equation, or Jourdan's principle)

Newton–Euler Equation

Consider the rigid body in Fig. 7.1 whose center of mass is measured by the vector $\mathbf{r}_c$ in some fixed coordinate system. The velocity and acceleration of the center of mass are given by

$$\dot{\mathbf{r}}_c = \mathbf{v}_c, \qquad \dot{\mathbf{v}}_c = \mathbf{a}_c \tag{7.3}$$

The "over dot" represents a total derivative with respect to time. We represent the total sum of vector forces on the body from both mechanical and electromagnetic sources by **F**. Newton's law for the motion

FIGURE 7.4 Magnetically levitated rigid body (HSST MagLev prototype vehicle, 1998, Nagoya, Japan).

of the center of mass of a body with mass m is given by

$$m\dot{\mathbf{v}}_c = \mathbf{F} \tag{7.4}$$

If $\mathbf{r}$ is a vector to some point in the rigid body, we define a local position vector ρ by $\mathbf{r}_P = \mathbf{r}_c + \rho$. If a force $\mathbf{F}_i$ acts at a point $\mathbf{r}_i$ in a rigid body, then we define the moment of the force $\mathbf{M}$ about the fixed origin by

$$\mathbf{M}_i = \mathbf{r}_i \times \mathbf{F}_i \tag{7.5}$$

The total force moment is then given by the sum over all the applied forces as the body

$$\mathbf{M} = \sum \mathbf{r}_i \times \mathbf{F}_i = \mathbf{r}_c \times \mathbf{F} + \mathbf{M}_c \quad \text{where} \quad \mathbf{M}_c = \sum \rho_i \times \mathbf{F}_i \tag{7.6}$$

We also define the *angular momentum* of the rigid body by the product of a symmetric matrix of second moments of mass called the *inertia matrix* $\mathbf{I}_c$. The angular momentum vector about the center of mass is defined by

$$\mathbf{H}_c = \mathbf{I}_c \cdot \omega \tag{7.7}$$

Since $\mathbf{I}_c$ is a symmetric matrix, it can be diagonalized with principal inertias (or eigenvalues) $\{I_{ic}\}$ about principal directions (eigenvectors) $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In these coordinates, which are attached to the body, the angular momentum about the center of mass becomes

$$\mathbf{H}_c = I_{1c}\omega_1\mathbf{e}_1 + I_{2c}\omega_2\mathbf{e}_2 + I_{3c}\omega_3\mathbf{e}_3 \tag{7.8}$$

where the angular velocity vector is written in terms of principal eigenvectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ attached to the rigid body.

Euler's extension of Newton's law for a rigid body is then given by

$$\dot{\mathbf{H}}_c = \mathbf{M}_c \tag{7.9}$$

This equation says that the change in the angular momentum about the center of mass is equal to the total moment of all the forces about the center of mass. The equation can also be applied about a fixed point of rotation, which is not necessarily the center of mass, as in the example of the compound pendulum given below.

Equations (7.4) and (7.9) are known as the Newton–Euler equations of motion. Without constraints, they represent six coupled second order differential equations for the position of the center of mass and for the angular orientation of the rigid body.

Multibody Dynamics

In a serial link robot arm, as shown in Fig. 7.2, we have a set of connected rigid bodies. Each body is subject to both applied and constraint forces and moments. The dynamical equations of motion involve the solution of the Newton–Euler equations for each rigid link subject to the geometric or kinematics constraints between each of the bodies as in (7.2). The forces on each body will have applied terms $\mathbf{F}^a$, from actuators or external mechanical sources, and internal constraint forces $\mathbf{F}^c$. When friction is absent, the work done by these constraint forces is zero. This property can be used to write equations of motion in terms of scalar energy functions, known as Lagrange's equations (see below).

Whatever the method used to derive the equation of motions, the dynamical equations of motion for multibody systems in terms of generalized coordinates $\{q_k(t)\}$ have the form

$$\sum m_{ij}\ddot{q}_j + \sum\sum \mu_{ijk}\dot{q}_j\dot{q}_k = Q_i \tag{7.10}$$

The first term on the left involves a generalized symmetric mass matrix $m_{ij} = m_{ji}$. The second term includes Coriolis and centripetal acceleration. The right-hand side includes all the force and control terms. This equation has a quadratic nonlinearity in the generalized velocities. These quadratic terms usually drop out for rigid body problems with a single axis of rotation. However, the nonlinear inertia terms generally appear in problems with simultaneous rotation about two or three axes as in multi-link robot arms (Fig. 7.2), gyroscope problems, and slewing momentum wheels in satellites.

In modern dynamic simulation software, called multibody codes, these equations are automatically derived and integrated once the user specifies the geometry, forces, and controls. Some of these codes are called ADAMS, DADS, Working Model, and NEWEUL. However, the designer must use caution as these codes are sometimes poor at modeling friction and impacts between bodies.

7.5 Simple Dynamic Models

Two simple examples of the application of the angular momentum law are now given. The first is for rigid body rotation about a single axis and the second has two axes of rotation.

Compound Pendulum

When a body is constrained to a single rotary degree of freedom and is acted on by the force of gravity as in Fig. 7.5, the equation of motion takes the form, where θ is the angle from the vertical,

$$I\vartheta - (m_1L_1 - m_2L_2)g\,\sin\theta = T(t) \tag{7.11}$$

where $T(t)$ is the applied torque, $I = m_1L_1^2 + m_2L_2^2$ is the moment of inertia (properly called the second moment of mass). The above equation is nonlinear in the sine function of the angle. In the case of small motions about $\theta = 0$, the equation becomes a linear differential equation and one can look for solutions of the form $\theta = A\,\cos\omega t$, when $T(t) = 0$. For this case the pendulum exhibits sinusoidal motion with

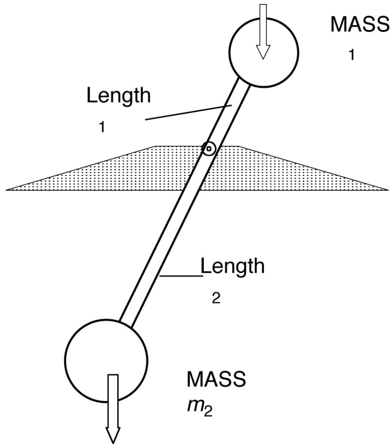

FIGURE 7.5 Sketch of a compound pendulum under gravity torques.

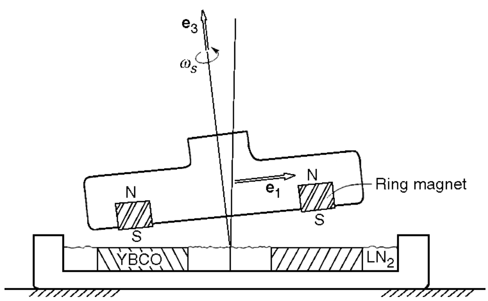

FIGURE 7.6 Sketch of a magnetically levitated flywheel on high-temperature superconducting bearings.

natural frequency

$$\omega = [g(m_2 L_2 - m_1 L_1)/I]^{1/2} \tag{7.12}$$

For the simple pendulum $m_1 = 0$, and we have the classic pendulum relation in which the natural frequency depends inversely on the square root of the length:

$$\omega = (g/L_2)^{1/2} \tag{7.13}$$

Gyroscopic Motions

Spinning devices such as high speed motors in robot arms or turbines in aircraft engines or magnetically levitated flywheels (Fig. 7.6) carry angular momentum, devoted by the vector **H**. Euler's extension of Newton's laws says that a change in angular momentum must be accompanied by a force moment **M**,

$$\mathbf{M} = \dot{\mathbf{H}} \tag{7.14}$$

In three-dimensional problems one can often have components of angular momentum about two different axes. This leads to a Coriolis acceleration that produces a gyroscopic moment even when the two angular motions are steady. Consider the spinning motor with spin ϕ about an axis with unit vector $\mathbf{e}_1$ and

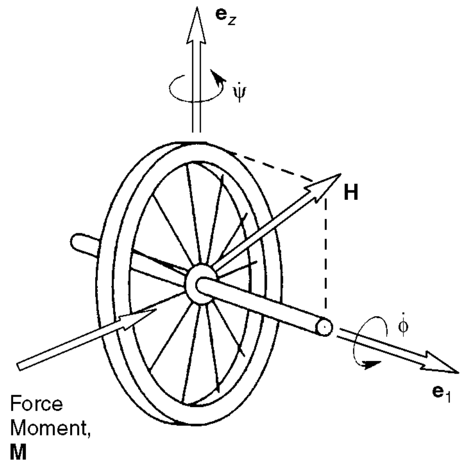

FIGURE 7.7 Gyroscopic moment on a precessing, spinning rigid body.

let us imagine an angular motion of the $\mathbf{e}_1$ axis, ψ about a perpendicular axis $\mathbf{e}_z$ called the precession axis in gyroscope parlance. Then one can show that the angular momentum is given by

$$\mathbf{H} = I_1\dot{\phi}\mathbf{e}_1 + I_z\dot{\psi}\mathbf{e}_z \tag{7.15}$$

and the rate of change of angular momentum for constant spin and presession rates is given by

$$\dot{\mathbf{H}} = \dot{\psi}\mathbf{e}_z \times \mathbf{H} \tag{7.16}$$

There must then exist a gyroscopic moment, often produced by forces on the bearings of the axel (Fig. 7.7). This moment is perpendicular to the plane formed by $\mathbf{e}_1$ and $\mathbf{e}_z$, and is proportional to the product of the rotation rates:

$$\mathbf{M} = I_1\dot{\phi}\dot{\psi}\mathbf{e}_z \times \mathbf{e}_1 \tag{7.17}$$

This has the same form as Eq. (7.10), when the generalized force Q is identified with the moment **M**, i.e., the moment is the product of generalized velocities when the second derivative acceleration terms are zero.

7.6 Elastic System Modeling

Elastic structures take the form of cables, beams, plates, shells, and frames. For linear problems one can use the method of eigenmodes to represent the dynamics with a finite set of modal amplitudes for generalized degrees of freedom. These eigenmodes are found as solutions to the PDEs of the elastic structure (see, e.g., Yu, 1996).

The simplest elastic structure after the cable is a one-dimensional beam shown in Fig. 7.8. For small motions we assume only transverse displacements $w(x, t)$, where x is a spatial coordinate along the beam. One usually assumes that the stresses on the beam cross section can be integrated to obtain stress vector resultants of shear V, bending moment M, and axial load T. The beam can be loaded with point or concentrated forces, end forces or moment or distributed forces as in the case of gravity, fluid forces, or electromagnetic forces. For a distributed transverse load $f(x, t)$, the equation of motion is given by

$$D\frac{\partial^4 w}{\partial x^4} - T\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = f(x, t) \tag{7.18}$$

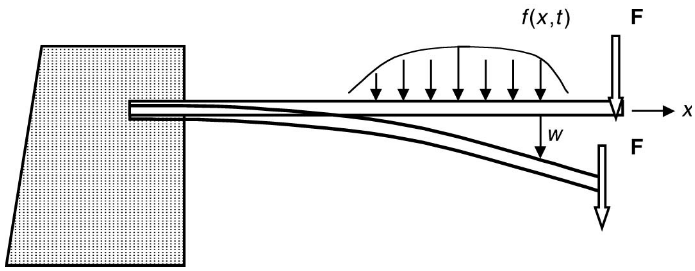

FIGURE 7.8 Sketch of an elastic cantilevered beam.

where D is the bending stiffness, A is the cross-sectional area of the beam, and ρ is the density. For a beam with Young's modulus Y, rectangular cross section of width b, and height h, $D = Ybh^3/12$. For $D = 0$, one has a cable or string under tension T, and the equation takes the form of the usual wave equation. For a beam with tension T, the natural frequencies are increased by the addition of the second term in the equation. For $T = -P$, i.e., a compressive load on the end of the beam, the curvature term leads to a decrease of natural frequency with increase of the compressive force P. If the lowest natural frequency goes to zero with increasing load P, the straight configuration of the beam becomes unstable or undergoes *buckling*. The use of T or $(-P)$ to stiffen or destiffen a beam structure can be used in design of sensors to create a sensor with variable resonance. This idea has been used in a MEMS accelerometer design (see below).

Another feature of the beam structure dynamics is the fact that unlike the string or cable, the frequencies of the natural modes are not commensurate due to the presence of the fourth-order derivative term in the equation. In wave type problems this is known as *wave dispersion*. This means that waves of different wavelengths travel at different speeds so that wave pulse shapes change their form as the wave moves through the structure.

In order to solve dynamic problems in finite length beam structures, one must specify boundary conditions at the ends. Examples of boundary conditions include

$$\begin{aligned}
&\text{clamped end} \quad w = 0, \quad \frac{\partial w}{\partial x} = 0 \\
&\text{pinned end} \quad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0 \text{ (zero moment)} \\
&\text{free end} \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^3 w}{\partial x^3} = 0 \text{ (zero shear)}
\end{aligned} \tag{7.19}$$

Piezoelastic Beam

Piezoelastic materials exhibit a coupling between strain and electric polarization or voltage. Thus, these materials can be used for sensors or actuators. They have been used for active vibration suppression in elastic structures. They have also been explored for active optics space applications. Many natural materials exhibit piezoelasticity such as quartz as well as manufactured materials such as barium titanate, lead zirconate titanate (PZT), and polyvinylidene fluoride (PVDF). Unlike forces on charges and currents (see below), the electric effect takes place through a change in shape of the material. The modeling of these devices can be done by modifying the equations for elastic structures.

The following work on piezo-benders is based on the work of Lee and Moon (1989) as summarized in Miu (1993). One of the popular configurations of a piezo actuator-sensor is the piezo-bender shown in Fig. 7.9. The elastic beam is of rectangular cross section as is the piezo element. The piezo element

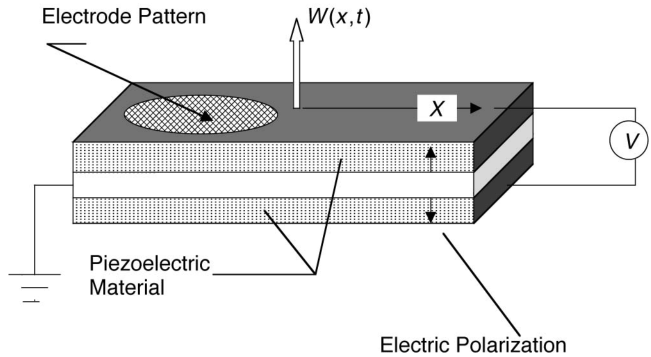

FIGURE 7.9 Elastic beam with two piezoelectric layers (Lee and Moon, 1989).

can be cemented on one or both sides of the beam either partially or totally covering the surface of the non-piezo substructure.

In general the local electric dipole polarization depends on the six independent strain components produced by normal and shear stresses. However, we will assume that the transverse voltage or polarization is coupled to the axial strain in the plate-shaped piezo layers. The constitutive relations between axial stress and strain, T, S, electric field and electric displacement, E_3, D_3 (not to be confused with the bending stiffness D), are given by

$$T_1 = c_{11}S_1 - e_{31}E_3, \qquad D_3 = e_{31}S_1 + \varepsilon_3 E_3 \tag{7.20}$$

The constants c_{11}, e_{31}, ε_3, are the elastic stiffness modulus, piezoelectric coupling constant, and the electric permittivity, respectively.

If the piezo layers are polled in the opposite directions, as shown in the Fig. 7.9, an applied voltage will produce a strain extention in one layer and a strain contraction in the other layer, which has the effect of an applied moment on the beam. The electrodes applied to the top and bottom layers of the piezo layers can also be shaped so that there can be a gradient in the average voltage across the beam width. For this case the equation of motion of the composite beam can be written in the form

$$D\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = -2e_{31}z_o\frac{\partial^2 V_3}{\partial x^2} \tag{7.21}$$

where $z_o = (h_S + h_P)/2$.

The z term is the average of piezo plate and substructure thicknesses. When the voltage is uniform, then the right-hand term results in an applied moment at the end of the beam proportional to the transverse voltage.

7.7 Electromagnetic Forces

One of the keys to modeling mechatronic systems is the identification of the electric and magnetic forces. Electric forces act on charges and electric polarization (electric dipoles). Magnetic forces act on electric currents and magnetic polarization. Electric charge and current can experience a force in a uniform electric or magnetic field; however, electric and magnetic dipoles will only produce a force in an electric or magnetic field gradient.

Electric and magnetic forces can also be calculated using both direct vector methods as well as from energy principles. One of the more popular methods is *Lagrange's equation* for electromechanical systems described below.

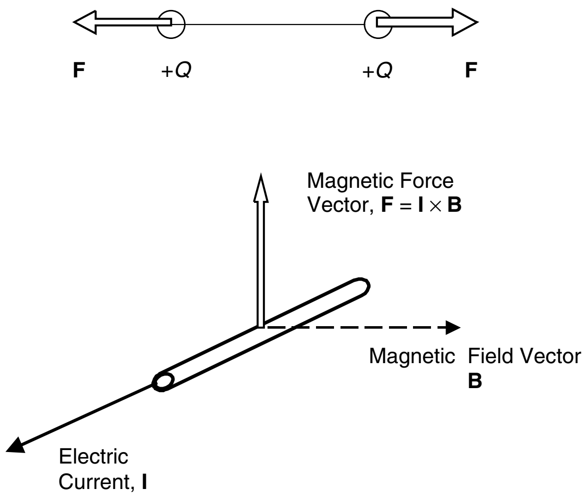

FIGURE 7.10 Electric forces on two charges (top). Magnetic force on a current carrying wire element (bottom).

Electromagnetic systems can be modeled as either distributed field quantities, such as electric field **E** or magnetic flux density **B** or as lumped element electric and magnetic circuits. The force on a point charge Q is given by the vector equation (Fig. 7.10):

$$\mathbf{F} = Q\mathbf{E} \tag{7.22}$$

When **E** is generated by a single charge, the force between charges Q_1 and Q_2 is given by

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r^2} \tag{7.23}$$

and is directed along the line connecting the two charges. Like charges repel and opposite charges attract one another.

The magnetic force per unit length on a current element **I** is given by the cross product

$$\mathbf{F} = \mathbf{I} \times \mathbf{B} \tag{7.24}$$

where the magnetic force is perpendicular to the plane of the current element and the magnetic field vector. The total force on a closed circuit in a uniform field can be shown to be zero. Net forces on closed circuits are produced by field gradients due to other current circuits or field sources.

Forces produced by field distributions around a volume containing electric charge or current can be calculated using the field quantities of **E**, **B** directly using the concept of magnetic and electric stresses, which was developed by Faraday and Maxwell. These electromagnetic stresses must be integrated over an area surrounding the charge or current distribution. For example, a solid containing a current distribution can experience a *magnetic pressure*, $P = B_t^2/2\mu_0$, on the surface element and a *magnetic tension*, $t_n = B_n^2/2\mu_0$, where the magnetic field components are written in terms of values tangential and normal to the surface. Thus, a one-tesla magnetic field outside of a solid will experience 40 N/cm^2 pressure if the field is tangential to the surface.

In general there are four principal methods to calculate electric and magnetic forces:

- direct force vectors and moments between electric charges, currents, and dipoles;
- electric field-charge and magnetic field-current force vectors;

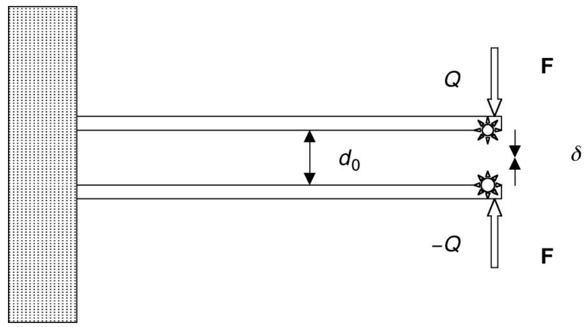

FIGURE 7.11 Two elastic beams with electric charges at the ends.

- electromagnetic tensor, integration of electric tension, magnetic pressure over the surface of a material body; and
- energy methods based on gradients of magnetic and electric energy.

Examples of the direct method and stress tensor method are given below. The energy method is described in the section on Lagrange's equations.

Example 1. Charge–Charge Forces

Suppose two elastic beams in a MEMS device have electric charges Q_1, Q_2 coulombs each concentrated at their tips (Fig. 7.11). The electric force between the charges is given by the vector

$$\mathbf{F} = \frac{Q_1 Q_2}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3} \quad \text{(newtons)} \tag{7.25}$$

where $1/4\pi\varepsilon_0 = 8.99 \times 10^9\ \text{Nm}^2/\text{C}^2$.

If the initial separation between the beams is d_0, we seek the new separation under the electric force. For simplicity, we let $Q_1 = -Q_2 = Q$, where opposite charges create an attractive force between the beam tips. The deflection of the cantilevers is given by

$$\delta = \frac{FL^3}{3YI} = \frac{1}{k}F \tag{7.26}$$

where L is the length, Y the Young's modulus, I the second moment of area, and k the effective spring constant.

Under the electric force, the new separation is $d = d_0 - 2\delta$,

$$k\delta = \frac{Q^2}{4\pi\varepsilon_0}\frac{1}{(d_0 - 2\delta)^2} \tag{7.27}$$

For $\delta \ll d_0$ to first order we have

$$\delta = \frac{Q^2/4\pi\varepsilon_0 d_0^2 k}{1 - (1/d_0^3)(Q^2/k\pi\varepsilon_0)} \tag{7.28}$$

This problem shows the potential for electric field buckling because as the beam tips move closer together, the attractive force between them increases. The nondimensional expression in the denominator

$$\frac{Q^2}{\pi\varepsilon_0 d_0^3}\frac{1}{k} \tag{7.29}$$

is the ratio of the negative electric stiffness to the elastic stiffness k of the beams.

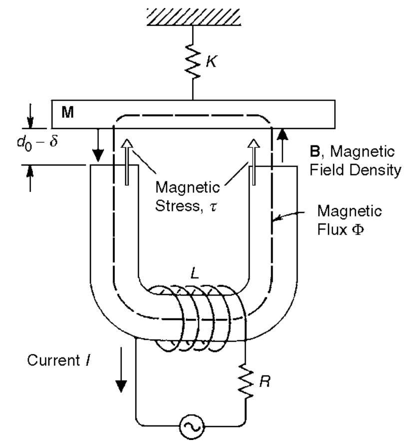

FIGURE 7.12 Force on a ferromagnetic bar near an electromagnet.

Example 2. Magnetic Force on an Electromagnet

Imagine a ferromagnetic keeper on an elastic restraint of stiffness k, as shown in Fig. 7.12. Under the soft magnetic keeper, we place an electromagnet which produces N turns of current I around a soft ferromagnetic core. The current is produced by a voltage in a circuit with resistance R.

The magnetic force will be calculated using the *magnetic stress tensor* developed by Maxwell and Faraday (see, e.g., Moon, 1984, 1994). Outside a ferromagnetic body, the stress tensor is given by **t** and the stress vector on the surface defined by normal **n** is given by $\boldsymbol{\tau} = \mathbf{t} \cdot \mathbf{n}$:

$$\tau = \frac{1}{\mu_0}\left(\frac{1}{2}[B_n^2 - B_t^2],\ B_n B_t\right) = (\tau_n, \tau_t) \tag{7.30}$$

For high magnetic permeability as in a ferromagnetic body, the tangential component of the magnetic field outside the surface is near zero. Thus the force is approximately normal to the surface and is found from the integral of the magnetic tension over the surface:

$$\mathbf{F} = \frac{1}{2\mu_0}\int B_n^2 \mathbf{n}\ dA \tag{7.31}$$

and $B_n^2/2\mu_0$ represents a magnetic tensile stress. Thus, if the area of the pole pieces of the electromagnet is A (neglecting fringing of the field), the force is

$$F = B_g^2 A/\mu_0 \tag{7.32}$$

where B_g is the gap field. The gap field is determined from Amperes law

$$NI = \widehat{R}\,\Phi, \quad \Phi = B_g A \tag{7.33}$$

where the *reluctance* is approximately given by

$$\widehat{R} = \frac{2(d_0 - \delta)}{\mu_0 A} \tag{7.34}$$

The balance of magnetic and elastic forces is then given by

$$F = \frac{1}{\mu_0 A}\Phi^2 = \frac{1}{\mu_0 A}\left(\frac{NI}{\hat{R}}\right)^2 = k\delta \tag{7.35}$$

or

$$\frac{(NI)^2}{4(d_0-\delta)^2}\mu_0 A = k\delta, \qquad \frac{\mu_0 N^2 I^2 A}{4(d_0-\delta)^2} = k\delta$$

(Note that the expression $\mu_0 N^2 I^2$ has units of force.) Again as the current is increased, the total elastic and electric stiffness goes to zero and one has the potential for buckling.

7.8 Dynamic Principles for Electric and Magnetic Circuits

The fundamental equations of electromagnetics stem from the work of nineteenth century scientists such as Faraday, Henry, and Maxwell. They take the form of partial differential equations in terms of the field quantities of electric field **E** and magnetic flux density **B**, and also involve volumetric measures of charge density q and current density **J** (see, e.g., Jackson, 1968). Most practical devices, however, can be modeled with lumped electric and magnetic circuits. The standard resistor, capacitor, inductor circuit shown in Fig. 7.13 uses electric current I (amperes), charge Q (columbs), magnetic flux Φ (webers), and voltage V (volts) as dynamic variables. The voltage is the integral of the electric field along a path:

$$V_{21} = \int_1^2 \mathbf{E} \cdot dl \tag{7.36}$$

The charge Q is the integral of charge density q over a volume, and electric current I is the integral of normal component of **J** across an area. The magnetic flux Φ is given as another surface integral of magnetic flux.

$$\Phi = \int \mathbf{B} \cdot d\mathbf{A} \tag{7.37}$$

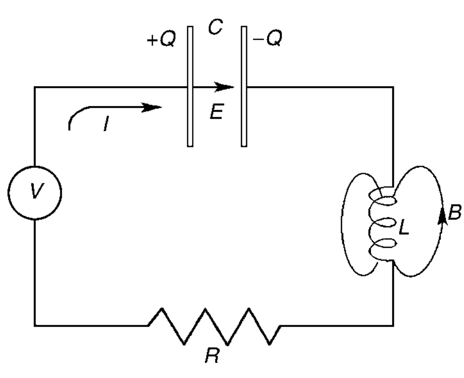

FIGURE 7.13 Electric circuit with lumped parameter capacitance, inductance, and resistance.

When there are no mechanical elements in the system, the dynamical equations take the form of conservation of charge and the Faraday–Henry law of flux change.

$$\frac{dQ}{dt} = I \quad \text{(Conservation of charge)} \tag{7.38}$$

$$\frac{d\phi}{dt} = V \quad \text{(Law of flux change)} \tag{7.39}$$

where $\phi = N\Phi$ is called the number of flux linkages, and N is an integer. In electromagnetic circuits the analog of mechanical constitutive properties is inductance L and capacitance C. The magnetic flux in an inductor, for example, often depends on the current I.

$$\phi = f(I) \tag{7.40}$$

For a linear inductor we have a definition of inductance L, i.e., $\phi = LI$. If the system has a mechanical state variable such as displacement x, as in a magnetic solenoid actuator, then L may be a function of x.

In charge storage circuit elements, the capacitance C is defined as

$$Q = CV \tag{7.41}$$

In MEMS devices and in microphones, the capacitance may also be a function of some generalized mechanical displacement variable.

The voltages across the different circuit elements can be active or passive. A pure voltage source can maintain a given voltage, but the current depends on the passive voltages across the different circuit elements as summarized in the Kirchhoff circuit law:

$$\frac{d}{dt}L(x)I + \frac{Q}{C(x)} + RI = V(t) \tag{7.42}$$

Lagrange's Equations of Motion for Electromechanical Systems

It is well known that the Newton–Euler equations of motion for mechanical systems can be derived using an energy principle called Lagrange's equation. In this method one identifies generalized coordinates $\{q_k\}$, not to be confused with electric charges, and writes the kinetic energy of the system T in terms of generalized velocities and coordinates, $T(\dot{q}_k, q_k)$. Next the mechanical forces are split into so-called conservative forces, which can be derived from a potential energy function $W(q_k)$ and the rest of the forces, which are represented by a generalized force Q_k corresponding to the work done by the kth generalized coordinate. Lagrange's equations for mechanical systems then take the form:

$$\frac{d}{dt}\frac{\partial T(\dot{q}_k, q_k)}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial W(q_k)}{\partial q_k} = Q_k \tag{7.43}$$

For example, in a linear spring–mass–damper system, with mass m, spring constant k, viscous damping constant c, and one generalized coordinate $q_1 = x$, the equation of motion can be derived using, $T = \frac{1}{2}m\dot{x}^2$, $W = \frac{1}{2}kx^2$, $Q_1 = -c\dot{x}$, in Lagrange's equation above. What is remarkable about this formulation is that it can be extended to treat both electromagnetic circuits and coupled electromechanical problems.

As an example of the application of Lagrange's equations to a coupled electromechanical problem, consider the one-dimensional mechanical device, shown in Fig. 7.14, with a magnetic actuator and a capacitance actuator driven by a circuit with applied voltage $V(t)$. We can extend Lagrange's equation to

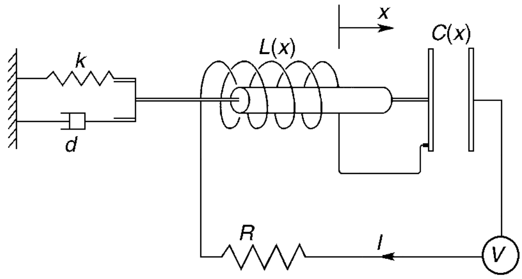

FIGURE 7.14 Coupled lumped parameter electromechanical system with single degree of freedom mechanical motion $x(t)$.

circuits by defining the charge on the capacitor, Q, as another generalized coordinate along with x, i.e., in Lagrange's formulation, $q_1 = x$, $q_2 = Q$. Then we add to the kinetic energy function a magnetic energy function $W_m(\dot{Q}, x)$, and add to the potential energy an electric field energy function $W_e(Q, x)$. The equations of both the mass and the circuit can then be derived from

$$\frac{d}{dt}\frac{\partial[T + W_m]}{\partial \dot{q}_k} - \frac{\partial[T + W_m]}{\partial q_k} + \frac{\partial[W + W_e]}{\partial q_k} = Q_k \tag{7.44}$$

The generalized force must also be modified to account for the energy dissipation in the resistor and the energy input of the applied voltage $V(t)$, i.e., $Q_1 = -c\dot{x}$, $Q_2 = -R\dot{Q} + V(t)$. In this example the magnetic energy is proportional to the inductance $L(x)$, and the electric energy function is inversely proportional to the capacitance $C(x)$. Applying Lagrange's equations automatically results in expressions for the magnetic and electric forces as derivatives of the magnetic and electric energy functions, respectively, i.e.,

$$W_m = \frac{1}{2}L(x)\dot{Q}^2 = \frac{1}{2}LI^2, \qquad W_e = \frac{1}{2C(x)}Q^2 \tag{7.45}$$

$$F_m = \frac{\partial W_m(x, \dot{Q})}{\partial x} = \frac{1}{2}I^2\frac{dL(x)}{dx}, \qquad F_e = -\frac{\partial W_e(x, Q)}{\partial x} = -\frac{1}{2}Q^2\frac{d}{dx}\left[\frac{1}{C(x)}\right] \tag{7.46}$$

These remarkable formulii are very useful in that one can calculate the electromagnetic forces by just knowing the dependence of the inductance and capacitance on the displacement x. These functions can often be found from electrical measurements of L and C.

Example: Electric Force on a Comb-Drive MEMS Actuator

Consider the motion of an elastically constrained plate between two grounded fixed plates as in a MEMS comb-drive actuator in Fig. 7.15. When the moveable plate has a voltage V applied, there is stored electric field energy in the two gaps given by

$$W_e^*(V, x) = \frac{1}{2}\varepsilon_0 V^2 A \frac{d_0}{d_0^2 - x^2} \tag{7.47}$$

In this expression the electric energy function is written in terms of the voltage V instead of the charge on the plates Q as in Eqs. (7.45) and (7.46). Also the initial gap is d_0, and the area of the plate is A.

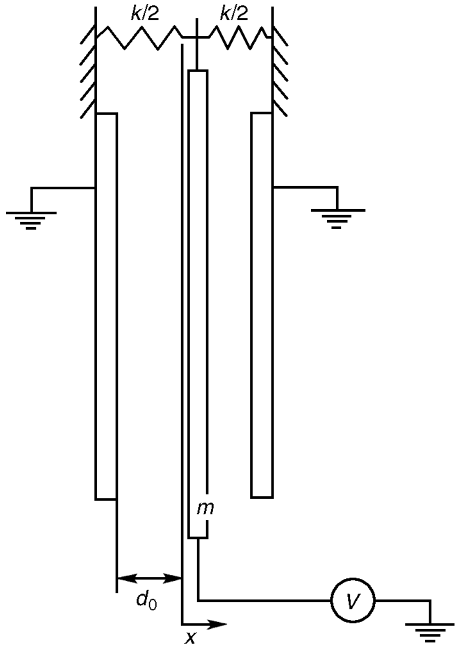

FIGURE 7.15 Example of electric force on the elements of a comb-drive actuator.

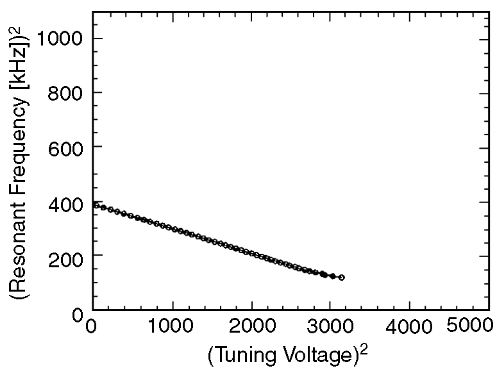

FIGURE 7.16 Decrease in natural frequency of a MEMS device with applied voltage as an example of negative electric stiffness [From Adams (1996)].

Using the force expressions derived from Lagrange's equations (7.44), the electric charge force on the plate is given by

$$F_e = \frac{\partial}{\partial x} W_e^*(V,\ x) = \frac{\varepsilon_0 V^2 A}{d_0} \frac{x}{(1 - x^2/d^2)^2} \tag{7.48}$$

This expression shows that the electric stiffness is negative for small x, which means that the voltage will decrease the natural frequency of the plate. This idea has been applied to a MEMS comb-drive actuator by Adams (1996) in which the voltage could be used to tune the natural frequency of a MEMS accelerometer, as shown in Fig. 7.16.

7.9 Earnshaw's Theorem and Electromechanical Stability

It is not well known that electric and magnetic forces in mechanical systems can produce static instability, otherwise known as *elastic buckling* or *divergence*. This is a consequence of the inverse square nature of many electric and magnetic forces. It is well known that the electric and magnetic field potential Φ satisfies Laplace's equation, $\nabla^2\Phi = 0$. There is a basic theorem in potential theory about the impossibility of a relative maximum or minimum value of a potential $\Phi(\mathbf{r})$ for solutions of Laplace's equation except at a boundary. It was stated in a theorem by Earnshaw (1829) that it is impossible for a static set of charges, magnetic and electric dipoles, and steady currents to be in a stable state of equilibrium without mechanical or other feedback or dynamic forces (see, for example, Moon, 1984, 1994).

One example of Earnshaw's theorem is the instability of a magnetic dipole (e.g., a permanent magnet) near a ferromagnetic surface (Fig. 7.17). Levitated bearings based on ferromagnetic forces, for example, require feedback control. Earnshaw's theorem also implies that if there is one degree of freedom with stable restoring forces, there must be another degree of freedom that is unstable. Thus the equilibrium positions for a pure electric or magnetic system of charges and dipoles must be saddle points. The implication for the force potentials is that the matrix of second derivatives is not positive definite. For example, suppose there are three generalized position coordinates $\{s_u\}$ for a set of electric charges. Then if the generalized forces are proportional to the gradient of the potential, $\nabla\Phi$, then the generalized electric stiffness matrix $\mathbf{K}_{ij}$, given by

$$\mathbf{K}_{ij} = \left[\frac{\partial^2\Phi}{\partial s_i \partial s_j}\right]$$

will not be positive definite. This means that at least one of the eigenvalues will have negative stiffness.

Another example of electric buckling is a beam in an electric field with charge induced by an electric field on two nearby stationary plates as in Fig. 7.15. The induced charge on the beam will be attracted to either of the two plates, but is resisted by the elastic stiffness of the beam. As the voltage is increased, the combined electric and elastic stiffnesses will decrease until the beam buckles to one or the other of the two sides. Before buckling, however, the natural frequency of the charged beam will decrease (Fig. 7.16). This property has been observed experimentally in a MEMS device. A similar magneto elastic buckling is observed for a thin ferromagnetic elastic beam in a static magnetic field (see Moon, 1984). Both electroelastic and magnetoelastic buckling are derived from the same principle of Earnshaw's theorem.

There are dramatic exceptions to Earnshaw's stability theorem. One of course is the levitation of 50-ton vehicles with magnetic fields, known as MagLev, or the suspension of gas pipeline rotors using feedback controlled magnetic bearings (see Moon, 1994). Here either the device uses feedback forces, i.e., the fields

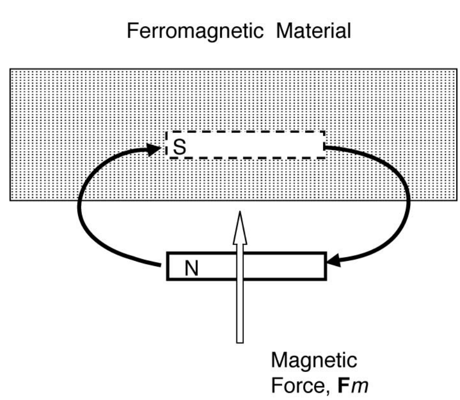

FIGURE 7.17 Magnetic force on a magnetic dipole magnet near a ferromagnetic half space with image dipole shown.

are not static, or the source of one of the magnetic fields is a superconductor. Diamagnetic forces are exceptions to Earnshaw's theorem, and superconducting materials have properties that behave like diamagnetic materials. Also new high-temperature superconductivity materials, such as YBaCuO, exhibit magnetic flux pinning forces that can be utilized for stable levitation in magnetic bearings without feedback (see Moon, 1994).

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