

Solve for $\frac{\partial p}{\partial x} = -\rho abx$ and $\frac{\partial p}{\partial y} = -\rho aby$, or: $\mathbf{p} = -\frac{\rho}{2} \mathbf{ab}(x^2 + y^2) + \text{const}$ Ans.

The given velocity field, $u = ay$ and $v = bx$, is an exact solution independent of a or b. It is not, however, an “irrotational” flow.

4.29 Consider a steady, two-dimensional, incompressible flow of a newtonian fluid with the velocity field $u = -2xy$, $v = y^2 - x^2$, and $w = 0$. (a) Does this flow satisfy conservation of mass? (b) Find the pressure field $p(x, y)$ if the pressure at point $(x = 0, y = 0)$ is equal to p_a .

Solution: Evaluate and check the incompressible continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = -2y + 2y + 0 \equiv 0 \quad \text{Yes!} \quad \text{Ans. (a)}$$

(b) Find the pressure gradients from the Navier-Stokes x - and y -relations:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad \text{or:}$$

$$\rho[-2xy(-2y) + (y^2 - x^2)(-2x)] = -\frac{\partial p}{\partial x} + \mu(0 + 0 + 0), \quad \text{or:} \quad \frac{\partial p}{\partial x} = -2\rho(xy^2 + x^3)$$

and, similarly for the y -momentum relation,

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \quad \text{or:}$$

$$\rho[-2xy(-2x) + (y^2 - x^2)(2y)] = -\frac{\partial p}{\partial y} + \mu(-2 + 2 + 0), \quad \text{or:} \quad \frac{\partial p}{\partial y} = -2\rho(x^2y + y^3)$$

The two gradients $\partial p/\partial x$ and $\partial p/\partial y$ may be integrated to find $p(x, y)$:

$$p = \int \frac{\partial p}{\partial x} dx|_{y=\text{const}} = -2\rho \left(\frac{x^2y^2}{2} + \frac{x^4}{4} \right) + f(y), \quad \text{then differentiate.}$$

$$\frac{\partial p}{\partial y} = -2\rho(x^2y) + \frac{df}{dy} = -2\rho(x^2y + y^3), \quad \text{whence} \quad \frac{df}{dy} = -2\rho y^3, \quad \text{or:} \quad f(y) = -\frac{\rho}{2}y^4 + C$$

$$\text{Thus:} \quad p = -\frac{\rho}{2}(2x^2y^2 + x^4 + y^4) + C = p_a \quad \text{at} \quad (x, y) = (0, 0), \quad \text{or:} \quad \mathbf{C} = \mathbf{p}_a$$

Finally, the pressure field for this flow is given by

$$\mathbf{p} = \mathbf{p}_a - \frac{1}{2}\rho(2x^2y^2 + x^4 + y^4) \quad \text{Ans. (b)}$$