This ODE has two types of solution, one very simple and one very complicated:

1. \( f = \text{constant}, \text{or: } v_r = \frac{\text{const}}{r} \) (a line source, as in Chap. 4)  \( \text{Ans.} \)

2. **Elliptic-integral solution** to the complete ODE above: these solutions, which vary in many ways with \( \theta \), represent “Jeffrey-Hamel” flow between plates.  \( \text{Ans.} \)

For further discussion of “Jeffrey-Hamel” flow, see pp. 168–172 of Ref. 5, Chap. 4.

4.34 A proposed three-dimensional incompressible flow field has the following vector form:

\[
V = Kx \hat{i} + Ky \hat{j} - 2Kz \hat{k}
\]

(a) Determine if this field is a valid solution to continuity and Navier-Stokes. (b) If \( g = -g \hat{k} \), find the pressure field \( p(x, y, z) \). (c) Is the flow irrotational?

**Solution:**  (a) Substitute this field into the three-dimensional incompressible continuity equation:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}(Kx) + \frac{\partial}{\partial y}(Ky) + \frac{\partial}{\partial z}(-2Kz)
\]

\[
= K + K - 2K = 0 \quad \text{Yes, satisfied} \quad \text{Ans. (a)}
\]

(b) Substitute into the full incompressible Navier-Stokes equation (4.38). The laborious results are:

- \( x \)-momentum: \( \rho(K^2x + 0 + 0) = -\frac{\partial p}{\partial x} + \mu(0 + 0 + 0) \)

- \( y \)-momentum: \( \rho(0 + K^2y + 0) = -\frac{\partial p}{\partial y} + \mu(0 + 0 + 0) \)

- \( z \)-momentum: \( \rho(0 + 0 + (-2Kz)(-2K)) = -\frac{\partial p}{\partial z} + \rho(-g) + \mu(0 + 0 + 0) \)

Integrate each equation for the pressure and collect terms. The result is

\[
p = p(0,0,0) - \rho gz - (\rho/2)K^2(x^2 + y^2 + 4z^2) \quad \text{Ans. (b)}
\]

Note that the last term is identical to \( (\rho/2)(u^2 + v^2 + w^2) \), in other words, Bernoulli’s equation.

(c) For irrotational flow, the curl of the velocity field must be zero:

\[
\nabla \times V = i(0 - 0) + j(0 - 0) + k(0 - 0) = 0 \quad \text{Yes, irrotational} \quad \text{Ans. (c)}
\]