This ODE has two types of solution, one very simple and one very complicated:

(1) f = constant, or: $\mathbf{v_r} = \dfrac{\mathbf{const}}{\mathbf{r}}$ (a line source, as in Chap. 4) *Ans.*

(2) **Elliptic-integral solution** to the complete ODE above: these solutions, which vary in many ways with θ, represent "Jeffrey-Hamel" flow between plates. *Ans.*

For further discussion of "Jeffrey-Hamel" flow, see pp. 168–172 of Ref. 5, Chap. 4.

4.34 A proposed three-dimensional incompressible flow field has the following vector form:

$$\mathbf{V} = Kx\mathbf{i} + Ky\mathbf{j} - 2Kz\mathbf{k}$$

(a) Determine if this field is a valid solution to continuity and Navier-Stokes. (b) If $\mathbf{g} = -g\mathbf{k}$, find the pressure field $p(x, y, z)$. (c) Is the flow irrotational?

Solution: (a) Substitute this field into the three-dimensional incompressible continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{\partial}{\partial x}(Kx)+\frac{\partial}{\partial y}(Ky)+\frac{\partial}{\partial z}(-2Kz)$$

$$= K + K - 2K = 0 \quad \textbf{Yes, satisfied}. \quad \textit{Ans. } \text{(a)}$$

(b) Substitute into the full incompressible Navier-Stokes equation (4.38). The laborious results are:

$$\text{x}-\text{momentum:} \quad \rho(K^2x+0+0) = -\frac{\partial p}{\partial x}+\mu(0+0+0)$$

$$\text{y}-\text{momentum:} \quad \rho(0+K^2y+0) = -\frac{\partial p}{\partial y}+\mu(0+0+0)$$

$$\text{z}-\text{momentum:} \quad \rho\{0+0+(-2Kz)(-2K)\} = -\frac{\partial p}{\partial z}+\rho(-g)+\mu(0+0+0)$$

Integrate each equation for the pressure and collect terms. The result is

$$\boldsymbol{p = p(0,0,0) - \rho g z - (\rho/2)K^2(x^2 + y^2 + 4z^2)} \quad \textit{Ans. } \text{(b)}$$

Note that the last term is identical to $(\rho/2)(u^2 + v^2 + w^2)$, in other words, Bernoulli's equation.

(c) For irrotational flow, the curl of the velocity field must be zero:

$$\nabla \times \mathbf{V} = \mathbf{i}(0-0) + \mathbf{j}(0-0) + \mathbf{k}(0-0) = \mathbf{0} \quad \textbf{Yes, irrotational}. \quad \textit{Ans. } \text{(c)}$$
