4.35 From the Navier-Stokes equations for incompressible flow in polar coordinates (App. E for cylindrical coordinates), find the most general case of purely circulating motion $\upsilon_\theta(r)$, $\upsilon_r = \upsilon_z = 0$, for flow with no slip between two fixed concentric cylinders, as in Fig. P4.35.

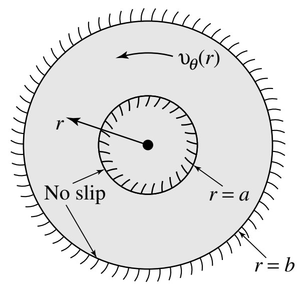

Fig. P4.35

Solution: The preliminary work for this problem is identical to Prob. 4.32 on the previous page. That is, there are two possible solutions for purely circulating motion υ_θ(r), hence

$$v_\theta = C_1 r + \frac{C_2}{r}, \quad \text{subject to } v_\theta(a) = 0 = C_1 a + C_2/a \quad \text{and} \quad v_\theta(b) = 0 = C_1 b + C_2/b$$

This requires $\mathbf{C_1 = C_2 = 0}$, or $\mathbf{v_\theta = 0}$ (no steady motion possible between fixed walls) *Ans.*

4.36 A constant-thickness film of viscous liquid flows in laminar motion down a plate inclined at angle θ, as in Fig. P4.36. The velocity profile is

$$u = Cy(2h - y) \quad v = w = 0$$

Find the constant C in terms of the specific weight and viscosity and the angle θ. Find the volume flux Q per unit width in terms of these parameters.

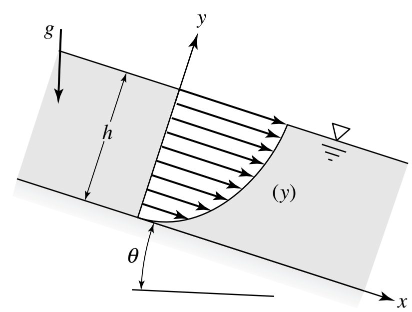

Fig. P4.36

Solution: There is atmospheric pressure all along the surface at y = h, hence $\partial p/\partial x = 0$. The x-momentum equation can easily be evaluated from the known velocity profile:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\nabla^2 u, \quad \text{or:} \quad 0 = 0 + \rho g \sin\theta + \mu(-2C)$$

$$\text{Solve for} \quad \mathbf{C = \frac{\rho g \sin\theta}{2\mu}} \quad \textit{Ans. (a)}$$

The flow rate per unit width is found by integrating the velocity profile and using C:

$$Q = \int_0^h u\,dy = \int_0^h Cy(2h - y)\,dy = \frac{2}{3}Ch^3 = \mathbf{\frac{\rho g h^3 \sin\theta}{3\mu}} \text{ per unit width} \quad \textit{Ans. (b)}$$

4.37 A viscous liquid of constant density and viscosity falls due to gravity between two parallel plates a distance 2h apart, as in the figure. The flow is fully developed, that is, w = w(x) only. There are no pressure gradients, only gravity. Set up and solve the Navier-Stokes equation for the velocity profile w(x).

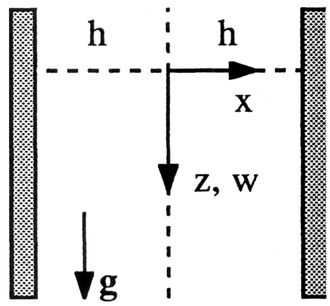

Fig. P4.37

Solution: Only the z-component of Navier-Stokes is relevant:

$$\rho\frac{dw}{dt} = 0 = \rho g + \mu\frac{d^2w}{dx^2}, \quad or: \quad w'' = -\frac{\rho g}{\mu}, \quad w(-h) = w(+h) = 0 \ \ (no\text{-}slip)$$

The solution is very similar to Eqs. (4.142) to (4.143) of the text:

$$w = \frac{\boldsymbol{\rho g}}{\boldsymbol{2\mu}}(\mathbf{h^2 - x^2}) \quad Ans.$$

4.38 Reconsider the angular-momentum balance of Fig. 4.5 by adding a concentrated *body couple* C_z about the *z* axis [6]. Determine a relation between the body couple and shear stress for equilibrium. What are the proper dimensions for C_z? (Body couples are important in continuous media with microstructure, such as granular materials.)

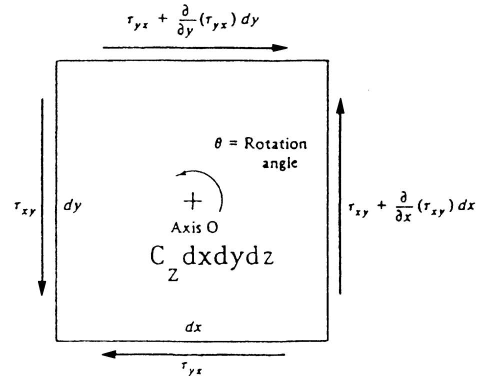

Fig. 4.5

Solution: The couple C_z has to be <u>per unit volume</u> to make physical sense in Eq. (4.39):

$$\left[\tau_{xy} - \tau_{yx} + \frac{1}{2}\frac{\partial\tau_{xy}}{\partial x}dx - \frac{1}{2}\frac{\partial\tau_{yx}}{\partial y}dy\right]dx\,dy\,dz + C_z dx\,dy\,dz = \frac{1}{12}\rho\,dx\,dy\,dz(dx^2 + dy^2)\frac{d^2\theta}{dt^2}$$

Reduce to third order terms and cancel (dx dy dz): $\boldsymbol{\tau_{yx} - \tau_{xy} = C_z}$ *Ans.*
The concentrated couple allows the stress tensor to have <u>unsymmetric</u> shear stress terms.
