Further show that the angular velocity about the z axis in such a flow would be given by

$$2\omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r\upsilon_\theta) - \frac{1}{r}\frac{\partial}{\partial \theta}(\upsilon_r)$$

Finally show that ϕ as defined satisfies Laplace's equation in polar coordinates for incompressible flow.

Solution: All of these things are quite true and easy to show from their definitions. *Ans.*

4.59 Consider the two-dimensional incompressible velocity potential $\phi = xy + x^2 - y^2$. (a) Is it true that $\nabla^2\phi = 0$, and, if so, what does this mean? (b) If it exists, find the stream function $\psi(x, y)$ of this flow. (c) Find the equation of the streamline which passes through $(x, y) = (2, 1)$.

Solution: (a) First check that $\nabla^2\phi = 0$, which means that **incompressible continuity is satisfied**.

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 + 2 - 2 = 0 \quad \textbf{Yes}$$

(b) Now use ϕ to find u and v and then integrate to find ψ.

$$u = \frac{\partial\phi}{\partial x} = y + 2x = \frac{\partial\psi}{\partial y}, \quad \text{hence} \quad \psi = \frac{y^2}{2} + 2xy + f(x)$$

$$v = \frac{\partial\phi}{\partial y} = x - 2y = -\frac{\partial\psi}{\partial x} = -2y - \frac{df}{dx}, \quad \text{hence} \quad f(x) = -\frac{x^2}{2} + const$$

The final stream function is thus $\boldsymbol{\psi = \frac{1}{2}(y^2 - x^2) + 2xy + const}$ *Ans.* (b)

(c) The streamline which passes through $(x, y) = (2, 1)$ is found by setting ψ = a constant:

$$At\ (x, y) = (2,\ 1), \quad \psi = \frac{1}{2}(1^2 - 2^2) + 2(2)(1) = -\frac{3}{2} + 4 = \frac{5}{2}$$

Thus the proper streamline is $\boldsymbol{\psi = \frac{1}{2}(y^2 - x^2) + 2xy = \frac{5}{2}}$ *Ans.* (c)
