CONSTANT COEF., HOMOG., LINEAR ODES
Complex Roots

Consider the ODE:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0 \qquad y(0) = \frac{1}{2} \quad y'(0) = 2$$

Following checks for order, coeff's, homog....
let $y = e^{mx}$
find Char. Eq^n is: $m^2 - 4m + 5 = 0$

* Does not factor nicely
* Use quadratic equation

$$m = \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$m = 2 \pm i \qquad \text{where } i = \sqrt{-1}$$

In this case, the LI roots are complex

$$y_1 = e^{(2+i)x} \qquad y_2 = e^{(2-i)x}$$

$$y = Ae^{(2+i)x} + Be^{(2-i)x} \qquad \text{(by superposition)}$$

<u>NOTES</u> - x and y are real
$\therefore$ A and B must be <u>complex</u>

- Hard to work with complex exponents, so use Euler's identity

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

The general solution is then:

$$y(x) = e^{2x}\left[A(\cos(x) + i\sin(x)) + B(\cos(x) - i\sin(x))\right]$$

$$y(x) = e^{2x}\left[(A+B)\cos(x) + i(A-B)\sin(x)\right]$$

REAL

$\therefore$ $(A+B) = C_1$ which is real

$i(A-B) = C_2$ which is ALSO real

From this,

$$y(x) = e^{2x}\left[C_1 \cos(x) + C_2 \sin(x)\right]$$

From the ICs

$$y(0) = \tfrac{1}{2} \longrightarrow C_1 = \tfrac{1}{2}$$

$$y'(0) = 2 = 2e^{2x}\left[C_1 \cos(x) + C_2 \sin(x)\right]\Big|_{x=0} + e^{2x}\left[-C_1 \sin(x) + C_2 \cos(x)\right]\Big|_{x=0}$$

$$\therefore 2 = 2\overset{1/2}{\cancel{C_1}} + C_2$$

$$C_2 = 2 - 1 = 1$$

The particular solution is

$$y(x) = e^{2x}\left[\tfrac{1}{2}\cos(x) + \sin(x)\right]$$

A NOTE ON COMPLEX ROOTS

Solutions to ODEs for which the Char. Eq^{n} has complex roots are often of the form:

$$y(x) = e^{ax}\left[c_1\cos(bx) + c_2\sin(bx)\right]$$

From $m = a \pm ib$

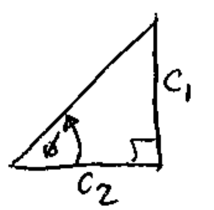
Using the identity:

$$\sin(x+\phi) = \sin(x)\underbrace{\cos(\phi)}_{c_2} + \cos(x)\underbrace{\sin(\phi)}_{c_1}$$

We can replace the above solution with

$$y(x) = Ke^{ax}\sin(bx+\phi)$$

STILL HAVE 2 CONSTANTS



$K = \sqrt{c_1^2 + c_2^2}$ — Simply the magnitude of the original $y(x) = e^{ax}[\text{———}]$

$\phi = \tan^{-1}\left(\frac{c_1}{c_2}\right)$

∴ From the previous example w. complex roots:

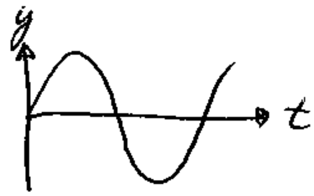
$$y = e^{2x}\left[\frac{1}{2}\cos(x) + \sin(x)\right]$$

$$y = \sqrt{\frac{5}{4}}\, e^{2x}\sin(x + 0.134\pi)$$

UNDAMPED OSCILLATIONS: The ODE

Oscillatory systems are very comon in all engineering disciplines; for this reason it is worth examing a theoretically ideal case, where there is not a damping effect. The corresponding ODE is:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$



Letting $y = e^{m}$, the characteristic eq^{n} becomes:

$$m^2 + \omega^2 = 0$$

$$m = \pm \omega i$$

The fundamental solutions are

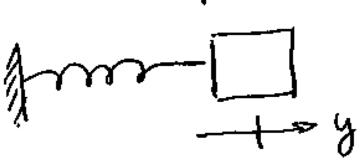
$$y_1 = \cos(\omega\ ) \qquad y_2 = \sin(\omega\ )$$

yielding the general solution

$$\boxed{y(x) = C_1 \cos(\omega t) + C_2 \sin(\omega t)}$$

RELATED TO PERIOD/FREQ.

Some particulars ....



$$ma = -ky$$

$$m\frac{d^2y}{dt^2} = -ky.$$

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

Thus, the frequency of oscillation is determined by $\sqrt{\frac{k}{m}}$!