Consider the 2nd order ODE, in standard form

$$\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)\,y = f(x)$$

There are 3 cases of missing terms to consider

i) $\frac{d^2y}{dx^2} = f(x)$ ie. $a_1(x) = a_0(x) = 0$

ii) $\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} = f(x)$ ie $a_0(x) = 0$

iii) $\frac{d^2y}{dx^2} + a_0(x)\,y = f(x)$ ie $a_1(x) = 0$

CASE 1

$$\frac{d^2y}{dx^2} = x \implies \frac{d}{dx}\left(\frac{dy}{dx}\right) = x$$

substitute $\boxed{v(x) = \frac{dy}{dx}}$

then $\frac{dv}{dx} = x \implies v(x) = \frac{x^2}{2} + C_1$

Recall, however, that:

$$v = \frac{dy}{dx} = \frac{x^2}{2} + C_1$$

Then $y(x) = \frac{1}{6}x^3 + C_1 x + C_2$

CASE ii)

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = 4x$$

Try $v(x) = \frac{dy}{dx}$ .....

$$\frac{dv}{dx} + \frac{1}{x}v = 4x$$

NOTE: This is now 1st Order Linear in $v(x)$.

Apply FOL formula...

$$v(x) = e^{-\int \frac{1}{x}dx}\left[\int 4x\, e^{\int \frac{1}{x}dx}\,dx + C_1\right]$$

$$= \frac{1}{x}\left[\int 4x^2\,dx + C_1\right]$$

$$v(x) = \frac{4}{3}x^2 + \frac{C_1}{x}$$

Now get $y(x)$....

$$v(x) = \frac{dy}{dx} = \frac{4}{3}x^2 + \frac{C_1}{x}$$

$$\boxed{y(x) = \frac{4}{9}x^3 + C_1 \ln x + C_2}$$

Some Notes:

→ 2nd Order ODE → 2 constants
  * 2 IC/BC required

→ $v(x) = \frac{dy}{dx}$ is a good substitution for LINEAR ODEs with NO term in $y(x)$ — ie. $a_0(x) = 0$

Case iii)

$$\frac{d^2y}{dx^2} + y = 0$$

NOTE: This ODE shows up in the solution process for many PDEs

Lets try $v(x) = \frac{dy}{dx}$ substitution, but the previous note suggests it may not work...

$$\frac{dv}{dx} + y = 0$$

Not too helpful, since we've now got an ODE in x, y and v!

Remember, however, that by the chain rule

$$\frac{dv}{dx} = \frac{dv}{dy} \overbrace{\frac{dy}{dx}}^{v}$$

Then, the ODE becomes:

$$v\frac{dv}{dy} + y = 0$$

SEPARABLE! But, things would be a lot harder if $f(x) \neq 0$, or with variable coefficients

Solve:

$$\int v\,dv = -\int y\,dy \Rightarrow v = \pm\sqrt{2c_1 - y^2}$$

★ Focus on $v > 0$

$$\frac{dy}{dx} = \left[2c_1 - y^2\right]^{1/2}$$

SEPARABLE

$$\int \frac{dy}{(2c_1 - y^2)^{1/2}} = \int dx$$

Use tables for $\int dy$

$$\sin^{-1}\left(y/\sqrt{2c_1}\right) = x + c_2$$

$$\boxed{y(x) = \sqrt{2c_1}\,\sin\left(x + c_2\right)}$$

What about higher order ODEs?

Yes, it is possible, BUT the chances of it working reduce with each order added, since certain lower order terms will need to be missing.

$$t^3 \dddot{y} + 3t^2 \ddot{y} = 1 \qquad \text{for} \quad t > 0$$

$\div t^3$ to get standard form.

$$\dddot{y} + \frac{3}{t}\ddot{y} = t^{-3} \qquad \text{make the substitution } w = \ddot{y}$$

$$\frac{dw}{dt} + \frac{3}{t} w = t^{-3} \qquad \text{FOL in } w(t)$$

$$\hookrightarrow w(t) = e^{\int -3/t \, dt}\left[\int t^{-3} e^{\int 3/t \, dt} \, dt + C_1\right]$$

$$= \frac{1}{t^2} + \frac{C_1}{t^3}$$

Now for $w = \ddot{y}$

$$\frac{d^2y}{dt^2} = \frac{1}{t^2} + \frac{C_1}{t^3} \qquad \text{use } v = \frac{dy}{dt} \ldots$$

$$\frac{dy}{dt} = -\frac{1}{t} - \frac{C_1}{2} t^{-2} + C_2$$

$$y(t) = -\ln(t) + \frac{C_1}{2} t^{-1} + C_2 t + C_3$$

☆ 3rd Order ODE

- 3 constants, 3 conditions for particular sol^{n}
- 3 linear ODEs must be solved