CONSTANT COEF., HOMOG., LINEAR ODEs

★ Real Roots, REPEATED

ODE

$$\left.\begin{aligned} y'' - 6y' + 9y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}\right\} \begin{array}{l} \bullet\ \text{C.C.} \\ \bullet\ \text{Homog} \\ \bullet\ \text{Linear} \\ \bullet\ \text{2nd Order} \end{array}$$

Recognize that $y = e^{mx}$ will be a sol^{n}:

* Subst: $m^2 \cancel{e^{mx}} - 6m\cancel{e^{mx}} - 9\cancel{e^{mx}}$

* Char. Eq^{n}: $m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

Roots are $m = 3,\ m = 3$ ← Real, but repeated

At this point, the solutions are $y = e^{3x}$ and $y = e^{3x}$

It should be VERY clear that these are NOT L.I.

– This means that only one of them is in the fundamental set of sol^{n}s ∴ we still need another....

Use D'Alembert!

Recall, let $y_2 = v(x)\, y_1 = v(x)\, e^{3x}$

$y_2' = v' e^{3x} + 3v e^{3x}$

$y_2'' = v'' e^{3x} + 3v' e^{3x} + 3v' e^{3x} + 9v e^{3x}$

Subst into ODE:

$$\left(v'' e^{3x} + \cancel{6v' e^{3x}} + \cancel{9v e^{3x}}\right) - \cancel{6v' e^{3x}} - \cancel{18v e^{3x}} + \cancel{9v e^{3x}} = 0$$

$v'' = 0 \longrightarrow v = Ax + B$ (by inspection)

$\therefore\ y_2 = (Ax + B)$

All we need is ONE LI solution to y_1, so choose
$A = 1,\ B = 0 \Rightarrow v = x \rightarrow y_2 = xe^{3x}$

2 Checks Are Possible (useful here)

1) Verify LI: Wronskian $\neq 0$ for any x — PASS
2) Verify y_2 satisfies the ODE ———— PASS

Since the ODE is linear, superposition applies:

$$y = C_1 e^{3x} + C_2 x e^{3x}$$

Now apply the ICs:

$y(0) = 1 \quad \therefore \quad C_1 = 1$

$y'(0) = 0 \quad \therefore \quad \left(3C_1 e^{3x} + C_2 e^{3x} + 3C_2 x e^{3x}\right|_{x=0} = 0$

$C_2 = -3$

The particular solution is then:

$$y(x) = e^{3x} - 3xe^{3x}$$

GENERAL OBSERVATIONS:

- for j repeated roots, the corresponding LI solutions are: $y_j = x^{j-1} e^{mx}$

Eg: 4 repeated roots of $(m-5)$
$y_1 = e^{mx},\ y_2 = xe^{mx},\ y_3 = x^2 e^{mx},\ y_4 = x^3 e^{mx}$
where $m = 5$