ODE SERIES SOLUTIONS
ALGEBRA OF SERIES

(Also see Boyce & DiPrima, Ch 5.1 in 7th Ed)

Summation Notation

Maclaurin Series:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n + \dots + a_\infty x^\infty$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

★ Counter (Index) Shift

$$y = \sum_{n=0}^{\infty} a_n x^n = \sum_{m=2}^{\infty} a_{m-2} x^{m-2}$$

Differentiation of Series

$$y = a_0 + a_1 x + a_2 x^2 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$= \frac{d}{dx} \sum a_n x^n$$

Bring the derivative "inside" the summation

$$= \sum_{n=0}^{\infty} a_n \frac{d}{dx}(x^n)$$

$$= \sum_{n=0}^{\infty} n\, a_n x^{(n-1)}$$

NOTICE: Term eliminated for $n=0$

Similarly

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

EQUATING POWER SERIES.

If $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, then $a_n = b_n$

Further, if $\sum_{n=0}^{\infty} a_n x^n = 0$, then $a_n = 0$

NOTE: Indices may need to be adjusted for the series to be equated, or, the series may only be equated over sub-ranges of the indices

SERIES SOLUTIONS OF ODES

Consider

$$y'' + y = 0, \quad y(x) = c_1 \cos(x) + c_2 \sin(x)$$

Recal, from the series representations of trig functions

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Thus, these two series are series sol$^{\underline{ns}}$ to the ODE, and when combined, they yield:

$$y(x) = c_1\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + c_2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)$$

While we will often be able to find a series solution, we will only know how it looks by plotting it