SERIES SOLUTION OF ODES

FROBENIUS SERIES EXAMPLE: BESSEL'S EQUATION

Bessel's Eq^n of Order "n" is:

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \qquad \text{(where } n = \text{const)}$$

* 2nd Order, linear, variable coef, homog.

Let $n = 0 \rightarrow$ Bessel Eq^n of Order "0"

$$x^2 y'' + xy' + x^2 y = 0$$

$$(\div x) \qquad x y'' + y' + xy = 0$$

Although we will do a series expansion about $x=0$, the value $x=0$ is now no longer valid!

$$p(x) = x \quad q(x) = 1 \quad r(x) = x$$

Thus, $p(0) = 0$ so $x=0$ is a singular point

- $\lim\limits_{x\to 0} \dfrac{x\,q(x)}{p(x)} = \lim\limits_{x\to 0} \dfrac{x}{x} = 1$ ✓
- $\lim\limits_{x\to 0} \dfrac{x^2 r(x)}{p(x)} = \lim\limits_{x\to 0} \dfrac{x^3}{x} = 0$ ✓

BOTH LIMITS EXIST $\therefore$ REGULAR SINGULAR POINT — FROBENIUS EXISTS

What about limits on convergence?

- $p(x) \neq 0$ for all other values of x; therefor the series converges for all x.

Now, subst $y = x^c \sum\limits_{n=0}^{\infty} a_n x^n = \sum\limits_{n=0}^{\infty} a_n x^{(n+c)}$

ODE:
$$\sum_{n=0}^{\infty} (n+c)(n+c-1) a_n x^{(n+c-1)}$$
$$+ \sum_{n=0}^{\infty} (n+c) a_n x^{(n+c-1)}$$
$$+ \sum_{n=0}^{\infty} a_n x^{(n+c+1)} = 0$$

The next step is to adjust the powers of x so that they are identical; for the first 2 terms,

$n+c-1 = m+c \quad \therefore m = n-1 \quad n = 1+m$

this is what we want for Frobenius

and for the third term:

$n+c+1 = m+c \quad \therefore m = n+1 \quad n = m-1$

The ODE then becomes (with the first 2 terms combined)

$$\sum_{m=-1}^{\infty} (m+c+1)^2 a_{m+1} x^{m+c} + \sum_{m=1}^{\infty} a_{m-1} x^{m+c} = 0$$

Note the start indexes are updated for $n=0$

We are now ready to combine the two terms, and attempt to find the $a_{m\pm1}$ constants. The first two terms from the first series are explicitly written and the remaining series are combined to yield

$$c^2 a_0 x^{c-1} + (c+1)^2 a_1 x^c + \sum_{m=1}^{\infty} \left[(m+c+1)^2 a_{m+1} + a_{m-1} \right] x^{m+c} = 0$$

Since this must be valid for all x,

① $c^2 a_0 = 0$

② $(c+1)^2 a_1 = 0$

③ $(m+c+1)^2 a_{m+1} + a_{m-1} = 0$

Look at these equalities in order $(a_0, a_1 \cdots)$

$\boxed{C^2 a_0 = 0}$

Since we do not know what the initial conditions are, we can not assume that $a_0 = 0$ (this would imply that $y(0) = x^c(0 + \cdots)$

eliminates this term

Thus, we arrive upon the condition that

$$c^2 = 0 \quad (\text{since } a_0 \neq 0)$$

★ This is called the INDICIAL EQUATION ★

NOTE: There is nothing particular about c^2. The indicial eq^{n} is defined as the "stuff" multiplying the a_0 term. For example, if ① were

$$(c)(c+1)\, a_0 = 0, \text{ then the}$$

I.E. would be $c(c+1) = 0$

This indicial eq^{n} has two identical roots, $c=0$

Then, using $c=0$, ② becomes

$$(\cancel{c}^{0}+1)^2 a_1 = 0 \quad \text{or} \quad a_1 = 0$$

At this point, a_0 is some arbitrary constant (from IC)

$a_1 = 0$

We still must find the remaing constants using a recursion relationship. From the series (3rd term)

$$a_{m+1} = \frac{-(a_{m-1})}{(m+\cancel{c}^{0}+1)^2} = \frac{-(a_{m-1})}{(m+1)}$$

$m=1: \quad a_2 = -\dfrac{a_0}{2^2}$

$m=2: \quad a_3 = \dfrac{-a_1}{3^2} = 0$

$m=3: \quad a_4 = \dfrac{-a_2}{4^2} = \dfrac{a_0}{2^2 4^2}$

etc ...

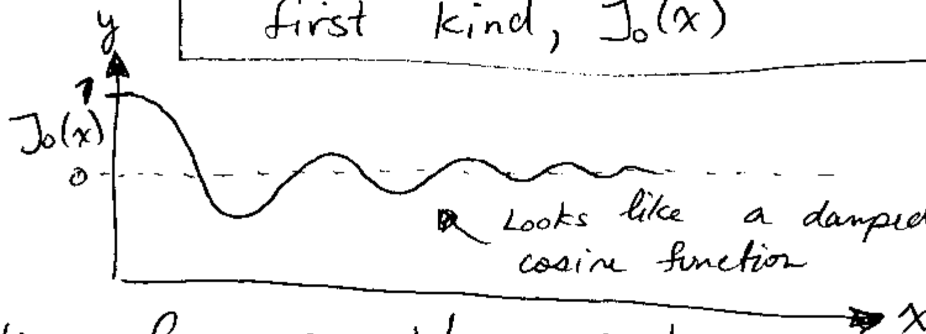
The series sol^n: $y(x) = x^c \{ a_0 + a_1 x + a_2 x^2 + \dots \}$

$$= x^0 \left\{ a_0 + 0 - \frac{a_0}{4} + 0 + \frac{a_0}{2^2 4^2} + \dots \right\}$$

$$y(x) = a_0 \underbrace{\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 4^2} - \frac{x^6}{2^2 4^2 6^2} + \dots \right)}$$

This series is common is certain fields of study (just like sine and cosine) and is given a name & is tabulated

$\boxed{\text{This is Bessel's Function of the first kind, } J_0(x)}$



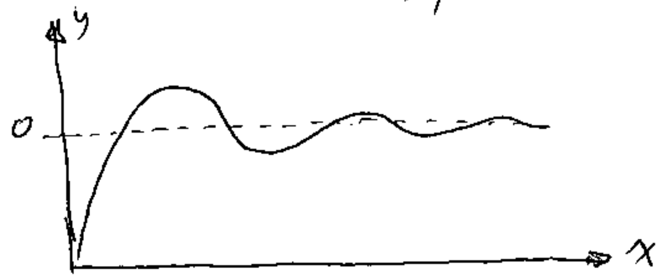
Notice, for our problem, we have
$y(x) = a_0 J_0$ ← there is only 1 arbitrary constant ?!

Our intuition about this second order ODE should tell use that another fundamental sol^n does exist. Indeed, if we trace back through the solution, there was one point where we arrived upon one, rather than two solutions... When we solved the INDICIAL EQUATION for the roots $c=0$ and $c=0$, the fact that they were identical meant that half of the complete solution would be eliminated.

The other fundamental solution $y_2(x)$ is given using D'Alembert, and is:

$$y_2(x) = J_0 \ln(x) + \frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 + \cdots$$

This is also a special equation, and is called "Bessel's function of the second kind, of zero order". It is represented as $Y_0(x)$, and is also tabulated.



Finally, the general solution to Bessel's eq^n of zero order is:

$$y(x) = C_1 J_0(x) + C_2 Y_0(x)$$