EXISTENCE & CONVERGENCE OF SERIES SOLUTIONS

The solution of Partial Differential Eqns (PDEs) often leads to ODEs of the general form:

$$p(x)y'' + q(x)y' + r(x)y = 0 \quad \left\{ \begin{array}{l} p, q, r \text{ are} \\ \text{specified polynomials} \\ \text{in } x \end{array} \right.$$

Specific examples include:

$$\left.\begin{array}{ll} \text{Cauchy Euler ODE} & c_2x^2y'' + c_1xy' + c_0y = 0 \end{array}\right\} \begin{array}{l}\text{Variable} \\ \text{Substitution}\end{array}$$

$$\left.\begin{array}{ll} \text{Bessel's ODE} & x^2y'' + xy' + (x^2-n^2)y = 0 \\ \text{Legendre ODE} & (1-x^2)y'' \; 2xy' + n(n+1)y = 0 \\ \text{Airy ODE} & y'' - xy = 0 \end{array}\right\} \begin{array}{l}\text{Use} \\ \text{Series} \\ \text{Solutions}\end{array}$$

General Form of Series Solutions

Consider the Taylor series $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

The choice of x_0 (ie the expansion point) is dictated by the ICs of the ODE. Since these are most monly set at $x=0$, the series reduces to a Maclaurin series, $y = \sum a_n x^n$

EXISTENCE & CONVERGENCE THEOREM

The existence and convergence of a series sol^{n} to the odes presented above are given by a theorem. For our purposes, the derivation/proof of the theorem is not necessary – we are only interested in the results...

Definitions

The point $x=0$ (ie expansion point) is always classified as one of the following:

1) $x=0$ is an ordinary point of the ODE if $p(0) \neq 0$

2) $x=0$ is a regular singular point of the ODE if $p(0)=0$, but

$$\lim_{x \to 0} \frac{x\,q(x)}{p(x)} \quad \text{AND} \quad \lim_{x \to 0} \frac{x^2 r(x)}{p(x)} \quad \text{both exist}$$

3) $x=0$ is an irregular singular point of the ODE if $p(0)=0$, and one (or both) of the limits from case #2 do not exist.

Theorem

1) If $x=0$ is an ordinary point, then a Maclaurin series solution $\left(y=\sum_{n=0}^{\infty} a_n x^n\right)$ always exists about the expansion point $x_0=0$.

2) If $x=0$ is a regular singular point, a Frobenius series solution $\left(y=x^c\sum_{n=0}^{\infty} a_n x^n\right)$ about $x=0$ allways exists.
   NOTE: "c" is a constant, possibly non-integer and/or negative

3) For 1 and 2, the series will converge for $|x|<R$, where R is the distance from the expansion point (ie $x=0$) to the nearest singular point of the ODE (ie, where $p(x)=0$)

4) If $x=0$ is an irregular singular point, then a series solution may or may not exist. Ultimately, a series solution should be tried, and convergence determined after-the-fact using the ratio test.