SERIES SOLUTIONS OF ODES
Maclaurin Series Example

Solve $y'' + 2xy' - y = 0$
ICs $y(0) = 1$
$y'(0) = 2$

- 2nd Order
- linear
- homog
- variable coefs.

Solution methodologies
- Reduction to 1ST order → not obvious
- Cauchy-Euler → NO!
- ▶ Series → Check $p(x)$ to see which series to use and if it will converge

In this case, $p(x) = 1$ ∴ $x = 0$ is an ordinary point and a Maclaurin series exists. Further, since $p(x) \neq 0$ for any x, then the series will converge for all x.

Now generate the series solution:
- assume $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$
$$= \sum_{n=0}^{\infty} a_n x^n$$

then
$$y'(x) = \sum n a_n x^{(n-1)}$$
$$y''(x) = \sum n(n-1) a_n x^{(n-2)}$$

* Our aim, is to substitute these series into the ode, and solve for the constants a_n

FIRST, subst y, y', y'' into the ODE

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2x\sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

SECOND, work with the series to combine them into a single series

- Bring $2x$ factor inside second term/series and combine it with the last term

$$\triangleright \quad 2x\sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} 2n a_n x^{(n-1+1)}$$

then the ODE becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} (2n-1) a_n x^n = 0$$

- Now adjust the indices to get x to the same power in each series, (ie x^m)

$$\blacktriangleright \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2} x^m$$

where $n-2=m \longrightarrow n=m+2$

$$\blacktriangleright \sum_{n=0}^{\infty} (2n-1)a_n x^n = \sum_{m=0}^{\infty} (2m-1) a_m x^m$$

$n=m \longrightarrow n=m$

- Finally, expand one of the series so that the start points are the same for the series

$$\blacktriangleright \sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2} x^m = \cancelto{0}{(0)(-1)a_0 x^{-2}} + \cancelto{0}{(1)(0)a_1 x^{-1}} + \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m$$

The resulting solution becomes:

$$\sum_{m=0}^{\infty}{}' (m+2)(m+1)\, a_{m+2}\, x^m + \sum_{m=0}^{\infty} (2m-1)\, a_m x^m = 0$$

and the single series is

$$\sum_{m=0}^{\infty}{}' \left[(m+2)(m+1) a_{m+2} + (2m-1) a_m\right] x^m = 0$$

THIRD, solve for the constants, a_m

Recall, that $\sum' b_m x^m = 0$ implies that $b_m = 0$

Thus, $(m+2)(m+1) a_{m+2} + (2m-1) a_m = 0$

a RECURSION RELATIONSHIP for the constants results, since

$$a_{m+2} = \frac{(1-2m)}{(m+2)(m+1)} \quad \text{for } m = 0, 1, 2, \ldots$$

Notice, that the constants a_0 and a_1 can-not be determined from this relationship – Indeed, these are 2 arbitrary constants for which the ICs are required. Furthermore, the

For a_0 & a_1 as arbitrary constants,

$m=0 \quad a_2 = \frac{1}{2} a_0$

$m=1 \quad a_3 = -\frac{1}{2\cdot3} a_1 = -\frac{1}{6} a_1$

$m=2 \quad a_4 = -\frac{3}{3\cdot4} a_2 = -\frac{1}{4} a_2 = -\frac{1}{8} a_0$

$m=3 \quad a_5 = \ldots$

Notice, that the $m =$ even are expressed in terms of a_0, and the odd constants are expressed in terms of a_1.

The final sol^n then becomes:

$$y = a_0\left(1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \tfrac{7}{210}x^6 + \ldots\right) + a_1\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{24}x^5 + \ldots\right)$$

The IC's are then applied to find the arbitrary constants a_0 & a_1. For $x=0$ ....

$y(0) = 1 \quad \therefore a_1 = 1$

$y'(0) = 2 \quad \therefore a_2 = 2.$