VARIABLE COEF ODES
Change of Independent Variable

Consider the Cauchy-Euler ODE

$$x^2 y'' + x y' + 4y = x \quad \left\{\begin{array}{l}\cdot \text{ Second Order} \\ \cdot \text{ linear} \\ \cdot \text{ variable coef.} \\ \cdot \text{ non-homogeneous}\end{array}\right.$$

Upon recognizing that this is a Cauchy Euler ODE, we may use the substitution $z = \ln(x)$, $x = e^z$

Then,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\left(\frac{1}{x}\right) = \frac{1}{e^z}\frac{dy}{dz} = e^{-z}\frac{dy}{dz}.$$

$$\begin{array}{c} y \\ | \\ z \\ | \\ x \end{array} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-z}\frac{dy}{dz}\right) = \frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \left(e^{-z}\frac{d^2y}{dz} - e^{-z}\frac{dy}{dz}\right)e^{-z}$$

$$= e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}.$$

Upon substitution into the ODE:

$$e^{2z}\left\{e^{-2z}\frac{d^2y}{dz^2} - e^{-2z}\frac{dy}{dz}\right\} + e^z\left\{e^{-z}\frac{dy}{dz}\right\} + 4y = e^z$$

the terms in $\frac{dy}{dx}$ cancel, leaving the CC ODE:

$$\frac{d^2y}{dz^2} + 4y = e^z$$

Solve $y'' + 4y = e^{z}$

let $y = y_c + y_p$.

<u>Complimentary</u>

$$\left.\begin{array}{l} m^2+4 \\ m=\pm 2i \end{array}\right\} y_c = C_1 \cos(2z) + C_2 \sin(2z)$$

<u>Particular</u>

Assume $y = a e^{z}$, solve to find $a = \frac{1}{5}$

<u>Final Sol^n</u>

$$y = C_1 \cos(2z) + C_2 \sin(2z) + \frac{1}{5} e^{z}$$

$$y(x) = C_1 \cos(2 \ln x) + C_2 \sin(2 \ln \ ) + \frac{1}{5} x.$$

<u>Some General Notes:</u>

- The Cauchy Euler Substitution works for any order ODE (provided it fits the form)
- Other substitutions work for other ODEs:

  $y'' - y' + e^{2x} y = 0$ try $z = e^{ax}$

  - Subst z into ODE, and choose "a" to simplify the ODE. The y' term is eliminated by choosing $a=1$ in this case