Consider the following problem with \( m = 3 \) constraints in \( n = 3 \) unknowns:

\[
\begin{align*}
\text{Maximize} & \quad x_1 - x_2 + x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + 2x_3 \leq 4 \\
& \quad 2x_1 - 3x_2 + x_3 \leq -5 \\
& \quad -x_1 + x_2 - 2x_3 \leq -1 \\
& \quad x_1, \ x_2, \ x_3 \geq 0
\end{align*}
\]

In \textbf{standard form} this becomes:

\[
\begin{align*}
\text{Maximize} & \quad x_1 - x_2 + x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + 2x_3 + x_4 = 4 \\
& \quad 2x_1 - 3x_2 + x_3 + x_5 = -5 \\
& \quad -x_1 + x_2 - 2x_3 + x_6 = -1 \\
& \quad x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6 \geq 0
\end{align*}
\]

On the previous handout (The Simplex Method Using Dictionaries) an initial BFS was obtained by making the original variables nonbasic (i.e., fixing their value to zero) and the slack variables basic. Using that same approach in this example would yield a basic solution \textit{that would be infeasible} (since \( x_5 = -5, \ x_6 = -1 \) violate their nonnegativity restrictions!). In such cases we create an artificial (Phase 1) problem, and its initial feasible dictionary, using the following \textbf{mutually exclusive and complimentary steps}:

1. start with the \textbf{standard form} LP (as above)

2. to each main constraint \( i \in \{1, ..., m\} \) \textbf{without} a slack variable, add (subtract) a unique \textbf{nonnegative artificial variable} to (from) the LHS if \( b_i \geq 0 \) \( (b_i < 0) \) and make this artificial variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.

3. to each main constraint \( i \in \{1, ..., m\} \) \textbf{with} a slack variable that would be \textbf{nonnegative if all original decision variables are set to zero}, make this slack variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.

4. to each main constraint \( i \in \{1, ..., m\} \) \textbf{with} a slack variable that would be \textbf{negative if all original decision variables are set to zero}, add (subtract) a unique \textbf{nonnegative artificial variable} to (from) the LHS if \( b_i \geq 0 \) \( (b_i < 0) \) and make this artificial variable basic in the initial feasible dictionary for the artificial (Phase 1) problem.

5. construct the equations for the Phase 1 initial feasible dictionary by isolating the basic variables (identified above) on the LHS of each main equation. The artificial objective, \( w \), to be \textbf{maximized} is the \textbf{negative sum} of the artificial variables that were introduced in steps 2 and 4 above. Include the artificial objective as the last equation in the initial feasible dictionary by isolating \( w \) on the LHS and by expressing \( w \) \textbf{in terms of the nonbasic variables (and a constant) only}. \textbf{Note: by construction,} \( w \leq 0 \) \textbf{for all feasible solutions}.

For this example we would construct the following artificial Phase 1 problem

\[
\begin{align*}
\text{Maximize} & \quad -x_7 - x_8 \\
\text{subject to} & \quad 2x_1 - x_2 + 2x_3 + x_4 = 4 \\
& \quad 2x_1 - 3x_2 + x_3 + x_5 - x_7 = -5 \\
& \quad -x_1 + x_2 - 2x_3 + x_6 - x_8 = -1 \\
& \quad x_1, \ x_2, \ x_3, \ x_4, \ x_5, \ x_6, \ x_7, \ x_8 \geq 0
\end{align*}
\]

and the corresponding initial dictionary

\[
\begin{align*}
x_4 &= 4 - 2x_1 + x_2 - 2x_3 \\
x_7 &= 5 + 2x_1 - 3x_2 + x_3 + x_5 \\
x_8 &= 1 - x_1 + x_2 - 2x_3 + x_6 \\
w &= -6 - x_1 + 2x_2 + x_3 - x_5 - x_6.
\end{align*}
\]
This initial dictionary is a feasible dictionary with:

Basic feasible solution \( x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 4, x_5 = 0, x_6 = 0, x_7 = 5, x_8 = 1, w = -6 \)

Entering variable \( x_2 \)
Leaving variable \( x_7 \)
Pivot equation \( x_2 = \frac{5}{3} + \frac{2}{5}x_1 + \frac{1}{5}x_3 + \frac{1}{5}x_5 - \frac{1}{3}x_7 \)

If we continue with the Simplex Method we obtain:

\[
\begin{align*}
x_2 &= \frac{5}{3} + \frac{2}{5}x_1 + \frac{1}{5}x_3 + \frac{1}{5}x_5 - \frac{1}{3}x_7 \\
x_4 &= \frac{17}{5} - \frac{2}{5}x_1 - \frac{1}{5}x_3 + \frac{1}{5}x_5 - \frac{1}{3}x_7 \\
x_8 &= \frac{8}{3} - \frac{1}{3}x_1 - \frac{5}{3}x_3 + \frac{1}{3}x_5 + x_6 - \frac{1}{3}x_7 \\
w &= -\frac{8}{3} + \frac{1}{3}x_1 + \frac{5}{3}x_3 - \frac{1}{3}x_5 - x_6 - \frac{2}{3}x_7
\end{align*}
\]

with

Basic feasible solution \( x_1 = 0, x_2 = \frac{5}{3}, x_3 = 0, x_4 = \frac{17}{5}, x_5 = 0, x_6 = 0, x_7 = 0, x_8 = \frac{8}{3}, w = -\frac{8}{3} \)

Entering variable \( x_3 \)
Leaving variable \( x_8 \)
Pivot equation \( x_3 = \frac{8}{5} - \frac{1}{5}x_1 + \frac{2}{5}x_5 + \frac{2}{5}x_6 - \frac{1}{5}x_7 - \frac{2}{5}x_8 \)

and finally

\[
\begin{align*}
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 - \frac{2}{5}x_7 - \frac{1}{5}x_8 \\
x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{2}{5}x_5 + \frac{3}{5}x_6 - \frac{1}{5}x_7 - \frac{2}{5}x_8 \\
x_4 &= 3 - x_1 - x_6 + x_8 \\
w &= 0 - x_7 - x_8
\end{align*}
\]

which is the optimal dictionary of this artificial Phase 1 problem with \( w^* = 0 \)

Since \( w^* = 0 \) this optimal dictionary for the artificial Phase 1 problem yields a feasible vertex of the original problem, namely \((x_1, x_2, x_3, x_4, x_5, x_6) = (0, \frac{11}{5}, \frac{8}{5}, 3, 0, 0)\). Furthermore, this Phase 1 dictionary can easily be converted into a corresponding feasible dictionary for the original problem by:

1. eliminating the artificial variables from all the optimal dictionary equations, and
2. replacing the equation for \( w \) with the original objective \( z \) expressed in terms of the nonbasic variables identified in the optimal artificial Phase 1 dictionary (ie. \( x_1, x_5 \) and \( x_6 \)).

Thus

\[
z = x_1 - x_2 + x_3 = x_1 - \left( \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \right) + \left( \frac{8}{5} - \frac{1}{5}x_1 + \frac{1}{5}x_5 + \frac{3}{5}x_6 \right) = \frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6
\]

and the initial dictionary for the original problem becomes:

\[
\begin{align*}
x_2 &= \frac{11}{5} + \frac{3}{5}x_1 + \frac{2}{5}x_5 + \frac{1}{5}x_6 \\
x_3 &= \frac{8}{5} - \frac{1}{5}x_1 + \frac{2}{5}x_5 + \frac{3}{5}x_6 \\
x_4 &= 3 - x_1 - x_6 \\
z &= \frac{3}{5} + \frac{1}{5}x_1 - \frac{1}{5}x_5 + \frac{2}{5}x_6
\end{align*}
\]

**Summary**

1. Construct the **artificial problem** and the corresponding initial feasible dictionary (as described above).
2. Solve the artificial problem via the Simplex Method.
3. If \( w^* = 0 \) transform this optimal artificial dictionary into an initial feasible dictionary for the original problem and proceed with the Simplex Method; otherwise, the original problem is infeasible.

This strategy is called the **TWO-PHASE SIMPLEX METHOD**.

In phase 1 we solve the artificial problem and in phase 2 we proceed with the original problem if possible.