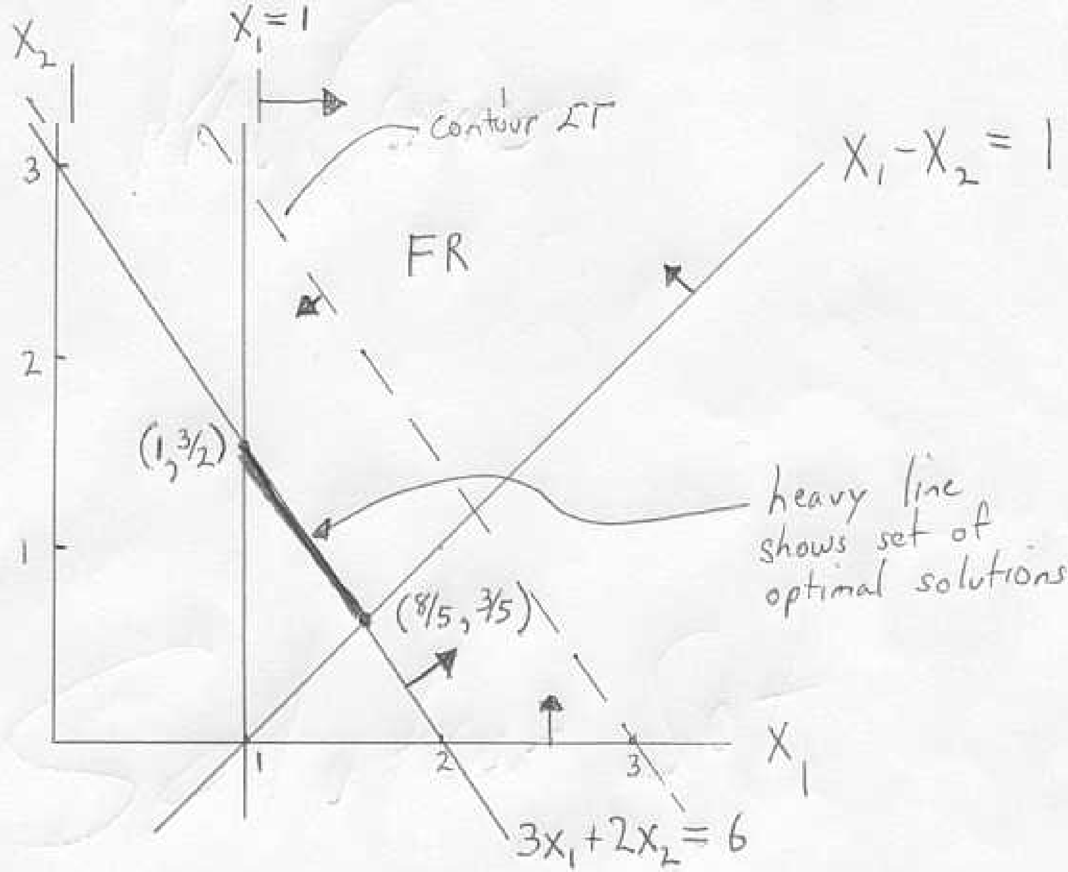
1. (Total 10%) Consider the linear program Model 1:

$$\begin{array}{llcrcl} \text{Minimize} & 9x_1 & + & 6x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & 1 \\ & 3x_1 & + & 2x_2 & \geq & 6 \\ & x_1 & & & \geq & 1 \\ \text{with } x_2 \geq 0 & & & & & \end{array}$$

(a) (5%) Solve Model 1 graphically. Be sure to clearly label constraints boundaries and objective function contours as well as the feasible region. Also indicate what solution(s) is(are) optimal.



(b) (5%) Write down the dual of Model 1 using variables v_1, v_2 and v_3 (corresponding to the three main primal constraints respectively).

$$\begin{array}{ll} \text{Maximize} & v_1 + 6v_2 + v_3 \\ \text{subject to} & v_1 + 3v_2 + v_3 = 9 \\ & -v_1 + 2v_2 \leq 6 \\ & v_1 \leq 0 \quad v_2 \geq 0 \quad v_3 \geq 0 \end{array}$$

2. (10%) In the interest of keeping its professional sports teams from being enticed away by neighbouring cities, Metropolis is considering construction of a new sports complex. There may be either a football or baseball stadium built, or both. Each stadium may also be either covered with a roof or open to the weather, with the roof either included with the original construction or added in the subsequent stage. The following table shows costs, in millions of dollars, of these different alternatives, depending on whether they are built in the first stage or the second, and depending on the priority the Metropolis Mayor places on them (100 being the highest).

	Football		Baseball	
	j=1:Stadium	j=2:Roof	j=3:Stadium	j=4:Roof
Cost if built in Stage 1	300	90	400	100
Cost if built in Stage 2	380	110	510	130
Priority	66	61	78	44

A total of $700 million is available in each stage to pay for the construction.

Formulate an integer linear program to choose a construction plan that maximizes total priority within budget limits and all needed logical constraints. Use only decision variables $x_j = 1$ if component j is built in stage 1 ($= 0$ otherwise), and $y_j = 1$ if component j is built in stage 2 ($= 0$ otherwise), $j = 1, ..., 4$. You may use symbolic parameter names for the constants in the above table, but clearly define any you employ. Also, briefly annotate all equations to indicate their meaning.

Let c_j^1 be the cost of building component j in Stage 1

Let c_j^2 be the cost of building component j in Stage 2

Let P_j be the priority of building component j

Maximize $\sum_{j=1}^{4} P_j (x_j + y_j)$ — total priority

s.t.

$\sum_{j=1}^{4} c_j^1 x_j \le 700$ — Stage 1 cost limit

$\sum_{j=1}^{4} c_j^2 y_j \le 700$ — Stage 2 cost limit

$x_j + y_j \le 1 \quad j=1,2,3,4$ — not in both stages

$x_2 \le x_1$ — no roof without stadium

$x_4 \le x_3$ — " " " "

$y_2 \le x_1 + y_1$ — " " " "

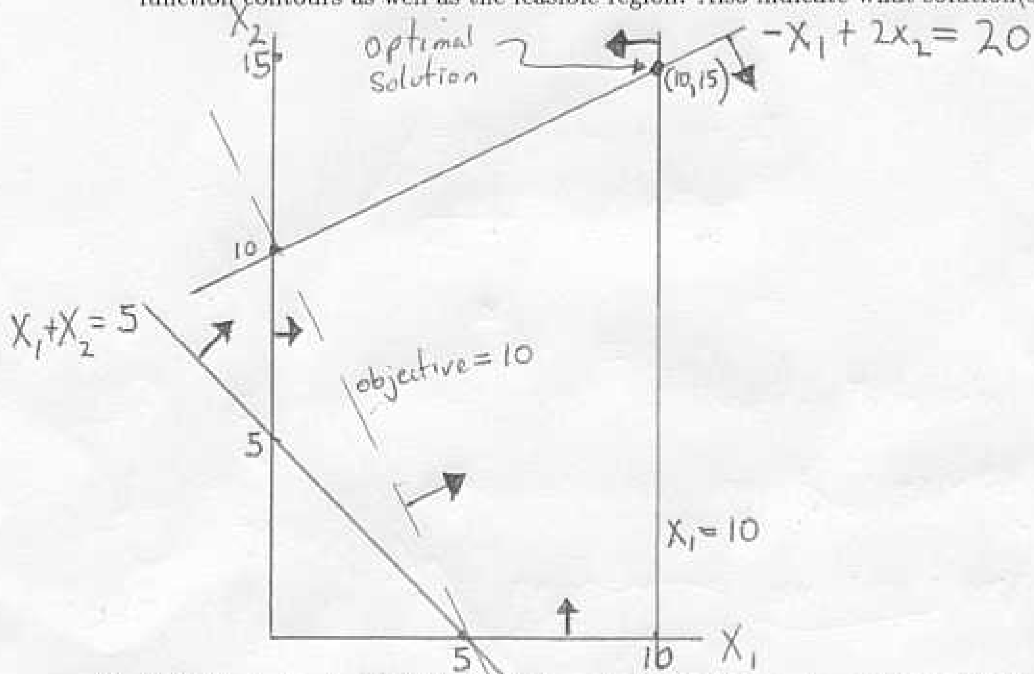
$y_4 \le x_3 + y_3$ — " " " "

$x_j, y_j = 1 \text{ or } 0 \quad j=1,2,3,4$ — binary variables

3. (Total 10%) Consider the linear program Model 2:

$$\begin{array}{lrcrcr} \text{Maximize} & 2x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \geq & 5 \\ & x_1 & & & \leq & 10 \\ & -x_1 & + & 2x_2 & \leq & 20 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

(a) (4%) Solve Model 2 graphically. Be sure to clearly label constraints boundaries and objective function contours as well as the feasible region. Also indicate what solution(s) is(are) optimal.



(b) (4%) State the dual of Model 2 using dual variables v_1, v_2, and v_3 (corresponding to the three main primal constraints respectively).

minimize $5v_1 + 10v_2 + 20v_3$

s.t. $v_1 + v_2 - v_3 \geq 2$

$v_1 + 2v_3 \geq 1$

$v_1 \leq 0 \quad v_2 \geq 0 \quad v_3 \geq 0$

(c) (2%) Without solving your dual from part (b), indicate which of its three dual variables must equal 0 at (dual) optimality as a consequence of your primal solution in part (a). Also briefly explain why.

$v_1^* = 0$ since the first primal constraint is not active at x^*

4. (Total 15%) Consider the following LP:

$$\begin{array}{lrcrcr} \text{Minimize} & 23x_1 & + & 12x_2 & & \\ \text{subject to} & 2x_1 & - & x_2 & \leq & 2 \\ & 4x_1 & - & x_2 & \leq & -5 \end{array}$$

where $x_1 \geq 0$ and $x_2 \geq 0$.

Apply Phase 1 of the Simplex method using Dictionaries to find an initial basic feasible solution and the corresponding initial feasible Phase 2 dictionary for this problem.

Begin by (a) putting the above problem into Standard Form. Then (b) construct the corresponding Phase 1 artificial (**maximization**) problem. Next (c) form the initial dictionary corresponding to this artificial (**maximization**) problem. Finally (d) solve, via dictionaries, this artificial (**maximization**) problem and use your solution to find an initial basic feasible solution and the corresponding initial feasible Phase 2 dictionary for this problem.

(Note: Please be careful – if you start with an incorrect initial Phase I maximization problem you can receive only half marks at best!)

Standard Form

Minimize $Z = 23X_1 + 12X_2$

s.t. $2X_1 - X_2 + X_3 = 2$ 3

$4X_1 - X_2 + X_4 = -5$

$X_1, X_2, X_3, X_4 \geq 0$

Phase 1 Artificial Problem

Maximize $W = -X_5$ 3

s.t. $2X_1 - X_2 + X_3 = 2$

$4X_1 - X_2 + X_4 - X_5 = -5$

$X_1, X_2, X_3, X_4, X_5 \geq 0$

Initial Phase 1 Dictionary (Maximization)

$$X_3 = 2 - 2x_1 + X_2$$
$$X_5 = 5 + 4x_1 - X_2 + X_4$$ 3
$$w = -5 - 4X_1 + X_2 - X_4$$

EBV $\quad x_2$

LBV $\quad x_5$

Pivot Eq $\quad X_2 = 5 + 4x_1 + X_4 - X_5$

$$X_3 = 7 + 2X_1 + X_4 - X_5$$ 3
$$X_2 = 5 + 4x_1 + X_4 - X_5$$
$$w = 0 - X_5$$

Initial Phase 2 BFS $\quad x = (0, 5, 7, 0)$

Initial Phase 2 Dictionary (Minimization)

$$X_3 = 7 + 2X_1 + X_4$$ 3
$$X_2 = 5 + 4x_1 + X_4$$

$$\begin{aligned} z &= 23x_1 + 12x_2 \\ &= 23x_1 + 12(5 + 4x_1 + x_4) \\ &= 60 + 71x_1 + 12x_4 \end{aligned}$$

Multiple Choice Section

17@2% each = Total 34%

5. In each of the following circle whichever choice provides the best answer.

The first few questions relate to the following Model 3:

$$
\begin{array}{llcll}
\text{Minimize} & \sum_{i=1}^{15}\sum_{j=1}^{300} c_{i,j} y_i x_{i,j} & & & \\
\text{subject to} & \sum_{i=1}^{15} y_i & = & 8 & \\
& \sum_{j=1}^{300} x_{i,j} & \leq & b_i, & i = 1, ..., 15 \\
& x_{i,j} & \leq & y_i & i = 1, ..., 15; j = 1, ..., 300 \\
& x_{i,j} & \geq & 0 & i = 1, ..., 15; j = 1, ..., 300 \\
& y_i & = & 0 \text{ or } 1 & i = 1, ..., 15
\end{array}
$$

(a) Model 3 is best described as:

(i) a linear program
(ii) a nonlinear program
(iii) an integer linear program
(iv) an integer nonlinear program

binary variables, nonlinear objective

(b) The **total** number of **main** constraints in Model 3 is:

(i) 3
(ii) 316
(iii) 4516
(iv) 9031

$1+15+15 \cdot 300$

(c) The **total** number of decision variables in Model 3 is:

(i) 2
(ii) 315
(iii) 4500
(iv) 4515

$15 \cdot 300$ X's + 15 y's

The next few questions relate to the following standard form linear program Model 4:

$$
\begin{array}{lcccccccc}
\text{Maximize} & 2x_1 & + & 5x_2 & + & 8x_3 & & \\
\text{subject to} & 2x_1 & + & x_2 & & & = & 5 \\
& x_1 & & & + & x_3 & = & 6 \\
& x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

(d) Which of the following is an interior point solution to Model 4?

strictly positive & feasible

(i) $x = (2, 1, 4)$
(ii) $x = (0, 5, 6)$
(iii) $x = (\frac{5}{2}, 0, \frac{7}{2})$
(iv) $x = (1, 3, 1)$

(e) Which of the following is an extreme point solution to Model 4?

(i) $x = (2, 1, 4)$
(ii) $x = (6, -7, 0)$
(iii) $x = (\frac{5}{2}, 0, \frac{7}{2})$
(iv) $x = (1, 3, 4)$

feasible and three activities (main constraints & restriction on X_2 active)

(f) Which of the following is **not** required of a feasible direction at $x = (\frac{5}{2}, 0, \frac{7}{2})$ in Model 4?

(i) $2\Delta x_1 + \Delta x_2 = 0$
(ii) $\Delta x_1 + \Delta x_3 = 0$
(iii) $\Delta x_2 \geq 0$
(iv) $\Delta x_3 \geq 0$

$X_3 \geq 0$ not active

(g) Which of the following **is** required of an improving direction at $x = (\frac{5}{2}, 0, \frac{7}{2})$ in Model 4?

(i) $2\Delta x_1 + \Delta x_2 = 0$
(ii) $2\Delta x_1 + 5\Delta x_2 + 8\Delta x_3 > 0$
(iii) $\Delta x_1 + \Delta x_3 = 0$
(iv) $2\Delta x_1 + 5\Delta x_2 + 8\Delta x_3 < 0$

$C \cdot \Delta X > 0$

The next few questions relate to the following minimizing standard form linear program Model 5:

	w_1	w_2	w_3	w_4	
min c	0	1	4	1	b
A	3	1	-2	1	5
	1	7	0	-1	3

(h) If w_2 and w_4 are basic in Model 5, the corresponding basic solution is:

$w_2 + w_4 = 5$
$7w_2 - w_4 = 3$

- (i) $w = (1, 4)$
- (ii) $w = (3, 0, 2, 0)$
- (iii) $w = (0, 1, 0, 4)$
- (iv) $w = (1, 1, 2, 5)$

(i) If w_1 and w_3 are basic in Model 5, the simplex direction for w_4 is:

$3\Delta w_1 - 2\Delta w_3 + 1 = 0$
$\Delta w_1 - 1 = 0$

- (i) $\Delta w = (1, 0, 2, 0)$
- (ii) $\Delta w = (1, 0, 2, 1)$
- (iii) $\Delta w = (-1, 0, -2, 1)$
- (iv) $\Delta w = (4, 0, 4, 1)$

(j) Direction $\Delta w = (1, -1, -2, -6)$ is a feasible dirction at solution $w = (1, \frac{2}{3}, \frac{2}{3}, \frac{8}{3})$ of Model 5. The maximum step λ that can be taken in this direction Δw is:

$\min\left\{\frac{2/3}{1}, \frac{2/3}{2}, \frac{8/3}{6}\right\}$

- (i) $\frac{1}{3}$
- (ii) $\frac{2}{3}$
- (iii) 1
- (iv) $\frac{3}{2}$

The next few questions relate to the following linear program Model 6:

$$
\begin{array}{lrcrcrcl}
\text{Maximize} & 18x_1 & - & 10x_2 & + & x_3 & & \\
\text{subject to} & 3x_1 & + & x_2 & + & 2x_3 & \leq & 9 \\
 & 2x_1 & - & 2x_2 & - & x_3 & \geq & 3 \\
 & x_1 & , & x_2 & , & x_3 & \geq & 0
\end{array}
$$

(k) What effect will an **increase** in the paramater 9 (RHS of first constraint) have on the Model 6 optimal solution value, assuming it has any effect at all?

relax + max
→ diminishing increase

- (i) increase at an acceleating rate
- (ii) increase at a diminishing rate
- (iii) decrease at an accelerating rate
- (iv) decrease at a diminishing rate

(l) What effect will an **increase** in the paramater 18 (objective coefficient of x_1) have on the Model 6 optimal solution value, assuming it has any effect at all?

max + increase in coef
→ accel increase

- (i) increase at an acceleating rate
- (ii) increase at a diminishing rate
- (iii) decrease at an accelerating rate
- (iv) decrease at a diminishing rate

(m) Assuming an optimal solution to the dual of Model 6 is $v^* = (6, 0)$, where v_1 and v_2 correspond to the first and second primal constraint respectively, which of the following is **not** implied about the primal optimum x^*?

$3v_1 + 2v_2 = 18 \Rightarrow$ dual active
$\Downarrow$
$x_1^* \geq 0$

- (i) primal optimal solution value is 54
- (ii) x^* makes the first constraint active
- (iii) $x_1^* = 0$
- (iv) $x_2^* = 0$

The next few questions relate to the following linear program Model 7:

$$\begin{array}{|lrcrcr|}\hline \text{Minimize} & x_1 & + & 3x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \geq & 5 \\ & x_1 & & & \leq & 4 \\ & & & x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \\ \hline \end{array}$$

(n) Which of the following **is** an optimal solution of Model 7?

(i) $(3, 2) \Rightarrow z = 9$ (iii) $(1, 4) \Rightarrow z = 13$

(ii) $(4, 1) \Rightarrow z = 7$ (iv) $(4, 4) \Rightarrow z = 16$

(o) Which of the following could be the **first** point of a Two-Phase Simplex search of Model 7?

(i) $(4, 4)$ BFS (iii) $(0, 0)$ infeasible

(ii) $(3, 2)$ boundary, not extreme (iv) $(5, 0)$ infeasible

(p) Assigning dual variables v_1, v_2, and v_3 to the 3 main constraints of Model 7 respectively, which of the following **is not** a primal complementary slackness condition on optimal values of x's and v's?

(i) $x_1 + x_2 = 5$ or $v_1 = 0$ (iii) $x_2 = 4$ or $v_3 = 0$

(ii) $x_1 = 4$ or $v_2 = 0$ (iv) $v_1 + v_3 = 3$ or $x_2 = 0$ ← this is dual complementary slackness

The final Multiple Choice Section question is:

(q) Which of the following is **not** an outcome on which the Two-Phase Simplex method would terminate?

(i) Phase 1 global optimum value $\neq 0$ (iii) Phase 1 global optimum value $= 0$ Continue to Phase II

(ii) Phase 2 global optimum value $\neq 0$ (iv) indication of unboundedness reached in Phase 2

True or False Section

5@2% each = Total 10%

6. Circle the correct answer to the following questions involving linear programming.

(a) True or False - A primal basic feasible solution is degenerate *if, and only if,* at least one of the primal nonbasic variables has a value of zero.

(b) True or False - LPs are deterministic models.

(c) True or False - The coefficient of the j^{th} decision variable in the i^{th} primal main constraint is also the coefficient of the i^{th} decision variable in the j^{th} dual main constraint.

(d) True or False - Every LP has an optimal solution that is a *basic feasible solution.* (Some LP have no solutions)

(e) True or False - A feasible solution x to an LP cannot be optimal if there exist any *improving* directions at x. (There can't be any improving feasible direction.)

Short Answer Section

(Total 10%)

7. Provide brief answers to the following questions involving LP.

(a) (4%) *Briefly* describe the four assumptions that all linear programs must satisfy.

All equations must satisfy the "proportionality assumption"
– if one unit of activity j consumes/produces a_{ij} units of resource/commodity i and yields c_j units of profit/cost then x_j units of activity j consumes/produces $a_{ij}x_j$ units and yields $c_j x_j$ units

All equations must satisfy the "additivity assumption"
– $f(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} f(x_i)$

All decision variables must satisfy the "divisibility assumption"
– activities can be conducted at any fractional level within bounds

All model parameters must satisfy the "certainty assumption"
– parameters are (assumed to be) constant (not stochastic)

(b) (2%) *Briefly* describe the significance of LP marginal values.

Marginal values indicate the rate of change of the primal objective value that would result if an extra unit of the corresponding RHS resource or commodity were available

(c) (2%) *Briefly* describe the condition(s) that improving directions must satisfy.

Δx is an improving direction at x if the objective function value at $x + \lambda \Delta x$ is superior to that at x for all $\lambda > 0$ sufficiently small $\Rightarrow$ $\nabla f(x) \cdot \Delta x < 0$ for minimization and $\nabla f(x) \cdot \Delta x > 0$ for maximization

(d) (2%) *Briefly* describe the condition(s) that feasible directions must satsify.

Δx is a feasible direction at x if the point $x + \lambda \Delta x$ is feasible if $\lambda > 0$ is sufficiently small

$\Rightarrow$ $\nabla g \cdot \Delta x \geq 0$ for all active $g(x) \geq b$

$\nabla g \cdot \Delta x \leq 0$ for all active $g(x) \leq b$

$\nabla g \cdot \Delta x = 0$ for all equality constraints $g(x) = b$