UNIVERSITY OF WATERLOO

Department of Systems Design Engineering

SD311 Engineering Optimization Spring 2003

INSTRUCTOR: Paul H. Calamai, DC-2623 (x3182) , phcalamai@solstice

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SOLUTION SET

PLEASE NOTE:

1. There are 8 problems to complete in order to obtain full marks.
2. The time limit for the examination is 2 hours.
3. Read over all of the questions before starting ... you will find that some problems are easier than others. The marking scheme is outlined in the chart below.
4. In addition to this title page there should be 11 pages to this exam. Pages 6 and 10 are intentionally blank.
5. Writing and drawing instruments are the only aids permitted.

GOOD LUCK

#1	10	
#2	10	
#3	10	
#4	10	
#5	15	
#6	10	
#7	15	
#8	15	
Total	95	

1. (Total 10%) A company makes 5 different products. Each product has to be processed on three types of machines. Labor cost is \$6/hour on machine types 1 and 2, and \$4/hour on machine type 3. Other data is given in the table below. Formulate the problem of maximizing the **net weekly profit** of this company as an LP:

	Minutes of machine time per unit of product					Machine time available/week
	1	2	3	4	5	
Machine type 1	12	7	8	10	7	149 hours
2	9	7	5	0	14	129 hours
3	7	8	5	4	3	1118 hours
Price/unit (\$)	17	15	16	13	14	
Raw material cost/unit (\$)	3	2	0.9	1.2	1	

Be sure to state any assumptions that you have to make!

Indices:

Let $j = 1, \ldots, 5$ indicate the product being considered.

Decision Variables:

Let x_j denote the number of units of product j manufactured each week, $j = 1$ to 5.

NOTE: defining the variables to be $x_{i,j}$ where i corresponds to the machine type is inappropriate since each product j "has to be processed" on all three machines!

Assumption:

Assume that we can produce fractional units (i.e. assume that x_j need not be integer valued).

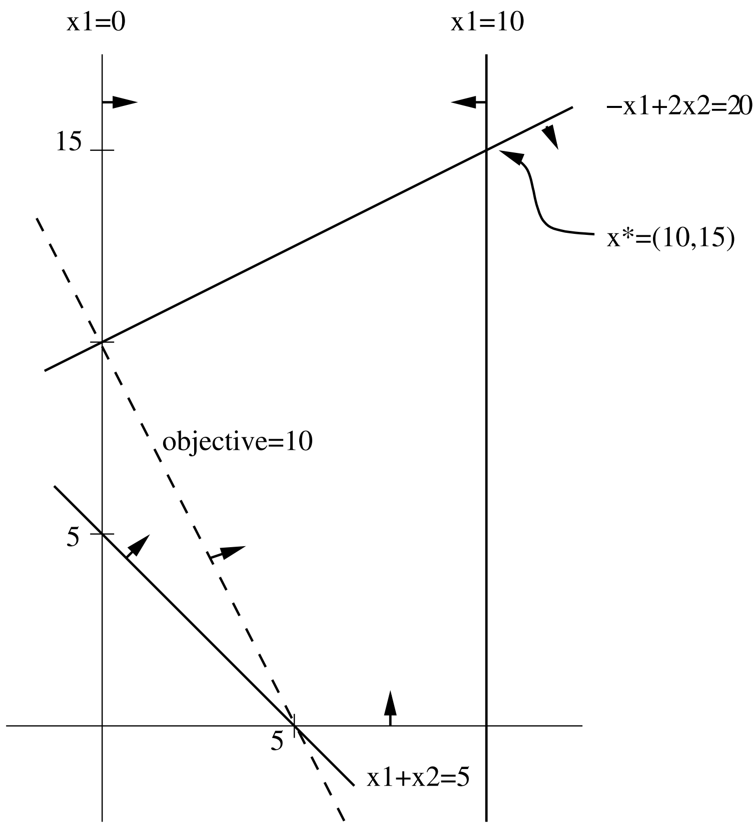
LP model:

$$\begin{array}{lll}
\text{Maximize} & (17-3)x_1 + (15-2)x_2 + (16-0.9)x_3 + (13-1.2)x_4 + (14-1)x_5 & (\text{\$ net weekly profit}) \\
& -(6/60)\left((12+9)x_1 + (7+7)x_2 + (8+5)x_3 + (10+0)x_4 + (7+14)x_5\right) & \\
& -(4/60)\left(7x_1 + 8x_2 + 5x_3 + 4x_4 + 3x_5\right) & \\
& & \\
\text{subject to} & 12x_1 + 7x_2 + 8x_3 + 10x_4 + 7x_5 \le 60 \times 149 & (\text{Machine 1 time limit, minutes}) \\
& 9x_1 + 7x_2 + 5x_3 + 0x_4 + 14x_5 \le 60 \times 129 & (\text{Machine 2 time limit, minutes}) \\
& 7x_1 + 8x_2 + 5x_3 + 4x_4 + 3x_5 \le 60 \times 1118 & (\text{Machine 3 time limit, minutes}) \\
& & \\
& x_1, x_2, x_3, x_4, x_5 \ge 0 & (\text{non-negativity})
\end{array}$$

2. (Total 10%) Solve the following linear program graphically using a scale of 1 inch = 5 units on both axes. Be sure to clearly label constraints boundaries and objective function contours as well as the feasible region. Also indicate what solution(s) is(are) optimal.

$$\begin{array}{lrcrcr} \text{Maximize} & 2x_1 & + & x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \geq & 5 \\ & x_1 & & & \leq & 10 \\ & -x_1 & + & 2x_2 & \leq & 20 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

Solution:



- you need to show the boundary of each constraint and somehow indicate where the feasible region is relative to each constraint
- you need to show at least one contour of the objective function and the direction of improvement of the objective
- you need to show the optimal solution

3. (Total 10%) Write the following LP in standard form as a **maximization** problem and then generate the corresponding standard display:

$$\begin{array}{ll} \text{Minimize} & 4x_1 + 2x_2 - 33x_3 \\ \text{subject to} & x_1 - 4x_3 + x_3 \leq 12 \\ & 9x_1 + 6x_3 = 15 \\ & -5x_1 + 9x_2 \geq 3 \\ & x_1 \leq 0, x_3 \geq 0 \end{array}$$

NOTE: You have to show and describe each step in the process.

Changing the sense of the objective yields:

$$\text{Maximize } -4x_1 - 2x_2 + 33x_3$$

Collecting terms on the left makes the first constraint:

$$x_1 - 3x_3 \leq 12$$

Introducing non-negative slack variables x_4 and x_5 in the first and third constraints respectively yields:

$$\begin{array}{l} x_1 - 3x_3 + x_4 = 12 \\ -5x_1 + 9x_2 - x_5 = 3 \end{array}$$

Replacing the nonpositive variable x_1 with $-x_6 = x_1$, and replacing unrestricted variable x_2 with the difference $x_2 = x_7 - x_8$ of non-negative variables, yields the following standard form representation of this LP:

$$\text{Maximize } 4x_6 - 2x_7 + 2x_8 + 33x_3$$

subject to

$$\begin{array}{rr} -x_6 - 3x_3 + x_4 = & 12 \\ -9x_6 + 6x_3 = & 15 \\ 5x_6 + 9x_7 - 9x_8 - x_5 = & 3 \end{array}$$

$$x_3, ..., x_8 \geq 0$$

The corresponding Standard Display is:

	x_3	x_4	x_5	x_6	x_7	x_8	
max c	33	0	0	4	-2	2	b
	-3	1	0	-1	0	0	12
	6	0	0	-9	0	0	15
	0	0	-1	5	9	-9	3
$BorN$							

4. (Total 10%) Consider the LP corresponding to the following Standard Display:

	x_1	x_2	x_3	x_4	
min c	0	1	10	0	b
	0	6	2	-2	2
	4	1	-1	21	11
	B	N	B	N	

(a) Compute the basic solution that corresponds to this display.

Setting nonbasic $x_2 = x_4 = 0$ and solving $Ax = b$ yields the BFS $x = (3, 0, 1, 0)$.

(b) Compute all Simplex directions that correspond to this display.

For x_2, we fix $\Delta x_2 = 1, \Delta x_4 = 0$ and solve $A\Delta x = 0$ to obtain $\Delta x = (-1, 1, -3, 0)$.

For x_4, we fix $\Delta x_2 = 0, \Delta x_4 = 1$ and solve $A\Delta x = 0$ to obtain $\Delta x = (-5, 0, 1, 1)$.

(c) Compute whether each Simplex direction is improving or not.

For x_2, we have $\bar{c} = c \cdot \Delta x = 1 - 30 = -29 < 0$ which tells us that Δx is improving (for minimization).

For x_4, we have $\bar{c} = c \cdot \Delta x = 10 > 0$ which tells us that Δx is not improving (for minimization).

(d) For any improving Simplex direction found above compute the corresponding maximum allowable step and specify the change of basis that would result if this step were taken.

Since $\Delta x = (-1, 1, -3, 0)$ is improving we have $\lambda = \min\left(\frac{x_1}{-\Delta x_1}, \frac{x_3}{-\Delta x_3}\right) = \min\left(\frac{3}{1}, \frac{1}{3}\right) = \frac{1}{3}$ is the maximum feasible step, and the new basis that would result after this step is $B = \{x_1, x_2\}$.

5. (Total 15%) Consider the following LP:

$$\begin{array}{lrcrcrcl} \text{Maximize} & 3x_1 & - & 10x_2 & + & 10x_3 & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & \leq & 2 \\ & x_1 & - & x_2 & + & x_3 & \leq & 1 \\ & 2x_1 & + & 20x_2 & + & x_3 & \leq & 5 \\ & & & -x_2 & + & x_3 & \leq & 0 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Begin by (a) putting the above problem into LP Standard Form. Then (b) form the initial **dictionary** corresponding to this LP. Next (c) perform, via dictionaries, two complete pivots (i.e. iterations) of the simplex method as described in your handouts (including all the corresponding information). Finally (d) interpret/classify the solution in your final dictionary.

(Note: Please be careful – if you start with an incorrect initial dictionary you can receive only half marks at best!)

NOTE: You have to show and describe each step in the process.

(a) Introduce non-negative slacks x_4, x_5, x_6, x_7 respectively into the four main constraints to obtain the following LP Standard Form:

$$\begin{array}{lrcrcrcccccl} \text{Maximize} & 3x_1 & - & 10x_2 & + & 10x_3 & & & & & & \\ \text{subject to} & -x_1 & + & x_2 & + & x_3 & + x_4 & & & & = & 2 \\ & x_1 & - & x_2 & + & x_3 & & + x_5 & & & = & 1 \\ & 2x_1 & + & 20x_2 & + & x_3 & & & + x_6 & & = & 5 \\ & & - & x_2 & + & x_3 & & & & + x_7 & = & 0 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$$

(b) For $B = \{x_4, x_5, x_6, x_7\}$ we get the following initial dictionary:

$$\begin{array}{rclcrcrcr} x_4 & = & 2 & + & x_1 & - & x_2 & - & x_3 \\ x_5 & = & 1 & - & x_1 & + & x_2 & - & x_3 \\ x_6 & = & 5 & - & 2x_1 & - & 20x_2 & - & x_3 \\ x_7 & = & 0 & & & + & x_2 & - & x_3 \\ & & & & & & & & \\ z & = & 0 & + & 3x_1 & - & 10x_2 & + & 10x_3 \end{array}$$

(c) For $EBV = \{x_3\}$, $LBV = \{x_7\}$, and pivot equation: $x_3 = x_2 - x_7$ we get the following dictionary:

$$\begin{array}{rclcrcrcr} x_4 & = & 2 & + & x_1 & - & 2x_2 & + & x_7 \\ x_5 & = & 1 & - & x_1 & & & + & x_7 \\ x_6 & = & 5 & - & 2x_1 & - & 21x_2 & + & x_7 \\ x_3 & = & 0 & & & + & x_2 & - & x_7 \\ & & & & & & & & \\ z & = & 0 & + & 3x_1 & & & - & 10x_7 \end{array}$$

with $B = \{x_3, x_4, x_5, x_6\}$.

Part (c) continued.

For $EBV = \{x_1\}$**,** $LBV = \{x_5\}$**, and pivot equation:** $x_1 = 1 - x_5 + x_7$ **we get the following dictionary:**

$$\begin{array}{rcrcrcrcr} x_4 & = & 3 & - & 2x_2 & - & x_5 & + & 2x_7 \\ x_1 & = & 1 & & & - & x_5 & + & x_7 \\ x_6 & = & 3 & - & 21x_2 & + & 2x_5 & - & x_7 \\ x_3 & = & 0 & + & x_2 & & & - & x_7 \\ z & = & 3 & & & - & 3x_5 & - & 7x_7 \end{array}$$

with $B = \{x_1, x_3, x_4, x_6\}$.

(d) This final dictionary corresponds to the BFS $x = (1, 0, , 0, 3, 0, 3, 0)$ **which is optimal, since the reduced cost** $\bar{c}$ **is non-positive for every nonbasic variable (**0 **for** x_2**,** -3 **for** x_5**,** -7 **for** x_7**) and this problem is a maximization LP, and degenerate since** x_3 **is basic and** $x_3 = 0$**. BONUS - this optimal solution is also non-unique since** $\bar{c}_2 = 0$**.**

> **NOTE: Do not confuse slack variables with artificial variables. The sum of the slack variables does not have to equal zero for the problem to be feasible!!!! The fact that you were able to start with an initial BFS means that this problem is feasible.**

6. (Total 10%) Suppose that an optimization model in decision variables x_1 and x_2 has constraints

$$\begin{aligned} 5x_1 + 2x_2 &\leq 10 \\ x_2 &\geq 0 \end{aligned}$$

(a) Devise a maximizing linear objective function for which the model has a unique optimal solution.

Maximize $z(x) = x_1 - x_2$ is one example. This is tricky since objectives like Maximize $z(x) = -x_2$ produce optimal solutions that are not unique. In general, any objective whose contours have a slope that is strictly between the slope of the first constraint boundary and the horizontal (i.e. the x_1 axis where $x_2 = 0$), and that improves to the right, will yield the unique solution $(x_1^*, x_2^*) = (2, 0)$. Such objectives take the form $ax_1 + bx_2$ where $a > \frac{5}{2}b$ when b is positive, or where a is positive when b is negative.

(b) Devise a maximizing linear objective function for which the model has multiple optimal solutions.

Maximize $z(x) = 5x_1 + 2x_2$ is one example and Maximize $z(x) = -x_2$ is another. Note that these two examples result in the extreme limits defining the range of the contour slope in the answer above!

(c) Devise a maximizing linear objective function for which the model is unbounded.

Maximize $z(x) = -x_1$ is one another example. In general, any objective whose contours have a slope that is strictly between the horizontal and the slope of the first constraint boundary, and that improves to the left, will result in a model that is unbounded. Note that this is the strict complement to the general answer given in part (a).

(d) Devise a minimizing linear objective function for which an interior point of this feasible region is optimal.

Minimize $z(x) = c$ where c is some constant is the only correct solution. This is the trivial case that was discussed in class for which any feasible solution is optimal.

(e) Devise a third linear constraint for which the model is infeasible.

$x_2 \leq -2$ **is one example.**

Multiple Choice problems, 6@2% + 1@3% = Total 15%

7. In each of the following circle whichever choice provides the best answer.

The first two questions relate to the following standard form linear program Model 1:

$$\begin{array}{lrcrcrcl} \text{Minimize} & 3x_1 & + & 2x_2 & + & 9x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & = & 9 \\ & x_1 & - & x_2 & & & = & 4 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

(a) Which of the following is an extreme point solution to Model 1?

(i) $x = (0, -4, 13)$ (iii) $x = (5, 1, 3)$

(ii) $x = (-1, -5, 15)$ **(iv)** $x = (\frac{13}{2}, \frac{5}{2}, 0)$

(b) Which of the following is required of a feasible Simplex direction, Δx, at $x = (4, 0, 5)$?

(i) $\Delta x_1 + \Delta x_2 + \Delta x_3 \geq 0$ (iii) $\Delta x_1 \geq 0$

(ii) $\Delta x_2 \geq 0$ (iv) $\Delta x_3 \geq 0$

The next two questions relate to the following maximizing standard form linear program Model 2:

	w_1	w_2	w_3	w_4	w_5	
max c	-1	3	0	7	-2	b
A	2	-1	1	4	1	8
	0	1	3	0	3	12

(c) If variables w_2 and w_4 are currently basic in Model 2, the simplex direction for w_5 in the next move in Simplex solution of Model 2 is:

(i) $\Delta w = (0, 0, 1, 0, 1)$ **(iii)** $\Delta w = (0, -3, 0, -1, 1)$

(ii) $\Delta w = (0, 0, -1, 0, 1)$ (iv) $\Delta w = (0, 9, 0, 4, 1)$

(d) The maximum feasible step α for a move in the direction $\Delta w = (-6, -1, 1, 0, 0)$ from solution $w = (6, 6, 1, 0, 1)$ of Model 2 is:

(i) 4 (iii) 2

(ii) 3 **(iv)** 1

The final three questions are True/False and Multiple Choice:

(e) If the feasible set of an optimization model is convex, there is a feasible direction leading from any feasible solution to any other feasible solution.

(i) **True** (ii) False

(f) A function f is unimodal if the straight line direction from every point in its domain to every better point in its domain is an improving direction.

(i) **True** (ii) False

(g) (3%) Suppose the two-phase improving search of a general mathematical program with original variables x_1, x_2, and x_3 terminates with the following Phase 1 solutions in different cases. Identify all cases that would proceed with Phase II.

(i) Global optimum $x = (22, 4, 3, 0, 0)$. (iii) Local optimum $x = (0, 2, 6, 5, 1)$, which may not be a global optimum

(ii) Global optimum $x = (40, 7, 0, 9, 0)$ **(iv)** Local optimum $x = (1, 3, 1, 0, 0)$.

8. (Total 15%) A company which makes two grades of industrial liquid cleaners labeled as L and H, is trying to revise their formulations. Each cleaner is made by mixing 4 different chemicals C_1 to C_4. The cleaners have to meet specifications on two different properties P_1, P_2 each measured in its own units. Relevant data is given below:

Chemical	Value of P_1	Value of P_2	Daily availability (barrels)	Cost per barrel ($)
C_1	12	150	1000	55
C_2	15	140	900	50
C_3	9	120	1500	45
C_4	20	155	1200	35

Cleaner	Max for P_1	Min for P_2	Exact demand (barrels/day)	Price per barrel ($)
L	15	142	2000	57
H	15	145	1300	62

Formulate the problem of finding the best blend for L and H, to maximize net daily profit, as an LP. Be sure to state any assumptions that you've made!

Indices:

Let $i = 1, 2$ represent the cleaners L and H respectively, and let $j = 1, \ldots, 4$ index the chemical C_j being considered.

Decision Variables:

Let x_{ij} denote the number of barrels of chemical C_j blended into cleaner i, $\forall i, j$.

Note that you might also define x_{ij} to be the proportion of chemical C_j in one barrel of cleaner i - but that this choice can result in nonlinear expressions if you are not careful with the resulting formulation.

Assumption:

Assume that we can use fractional barrels of any of the four chemicals (i.e. assume that x_{ij} need not be integer valued).

LP model:

We are required to maximize the net daily profit. The corresponding model is:

Maximize	$57\sum_{j=1}^{4} x_{1j} + 62\sum_{j=1}^{4} x_{2j}$	(\$ net daily profit)
	$-55\sum_{i=1}^{2} x_{i1} - 50\sum_{i=1}^{2} x_{i2} - 45\sum_{i=1}^{2} x_{i3} - 35\sum_{i=1}^{2} x_{i4}$	
subject to	$\sum_{j=1}^{4} x_{1j} = 2000$	(Demand for cleaner L)
	$\sum_{j=1}^{4} x_{2j} = 1300$	(Demand for cleaner H)
	$\sum_{i=1}^{2} x_{i1} \leq 1000$	(Supply of chemical 1)
	$\sum_{i=1}^{2} x_{i2} \leq 900$	(Supply of chemical 2)
	$\sum_{i=1}^{2} x_{i3} \leq 1500$	(Supply of chemical 3)
	$\sum_{i=1}^{2} x_{i4} \leq 1200$	(Supply of chemical 4)
	$12x_{11} + 15x_{12} + 9x_{13} + 20x_{14} - 15\sum_{j=1}^{4} x_{1j} \leq 0$	(Maximum P_1 in L)
	$12x_{21} + 15x_{22} + 9x_{23} + 20x_{24} - 15\sum_{j=1}^{4} x_{2j} \leq 0$	(Maximum P_1 in H)
	$150x_{11} + 140x_{12} + 120x_{13} + 155x_{14} - 142\sum_{j=1}^{4} x_{1j} \geq 0$	(Minimum P_2 in L)
	$150x_{21} + 140x_{22} + 120x_{23} + 155x_{24} - 145\sum_{j=1}^{4} x_{2j} \geq 0$	(Minimum P_2 in H)
	$x_{ij} \geq 0 \quad \forall i, j$	(non-negativity)