UNIVERSITY OF WATERLOO

Department of Systems Design Engineering

SD311 Engineering Optimization Spring 2002

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Mdterm Exam : Thursday June 13, 2002

SOLUTIONS by Paul Calamai

PLEASE NOTE:

1. There are 8 problems to complete in order to obtain full marks.
2. The time limit for the examination is 2 hours.
3. Read over all of the questions before starting ... you will find that some problems are easier than others. The marking scheme is outlined in the chart below.
4. In addition to this title page there should be 10 pages to this exam. Pages 6 and 10 are intentionally blank.
5. Drawing instruments are the only aids permitted.

GOOD LUCK

#1	10	
#2	10	
#3	10	
#4	10	
#5	15	
#6	10	
#7	15	
#8	20	
Total	100	

1. (Total 10%) A furniture manufacturer has 3600 board feet of Walnut, 4300 board feet of Maple, and 6550 board feet of Oak in stock. The manufacturer can produce three products using this lumber, with input requirements as given below:

	Wood needed (bd. ft./unit)			Profit
	Walnut	Maple	Oak	\$/unit
Table	10	50	100	100
Desk	10	30	40	50
Chair	80	5	0	10

A table is always sold in combination with 4 chairs; and a desk is always sold in combination with one chair. However, a chair can be sold independently. Formulate an LP model to determine the product mix that maximizes profit. Be sure to state any assumptions that you have to make!

Assumption:

We must assume that the variables are continuous for the formulation to be an LP.

Variables:

Let x_1, x_2 denote the number of tables and desks sold, respectively.

Let x_3 denote the number of chairs sold independently.

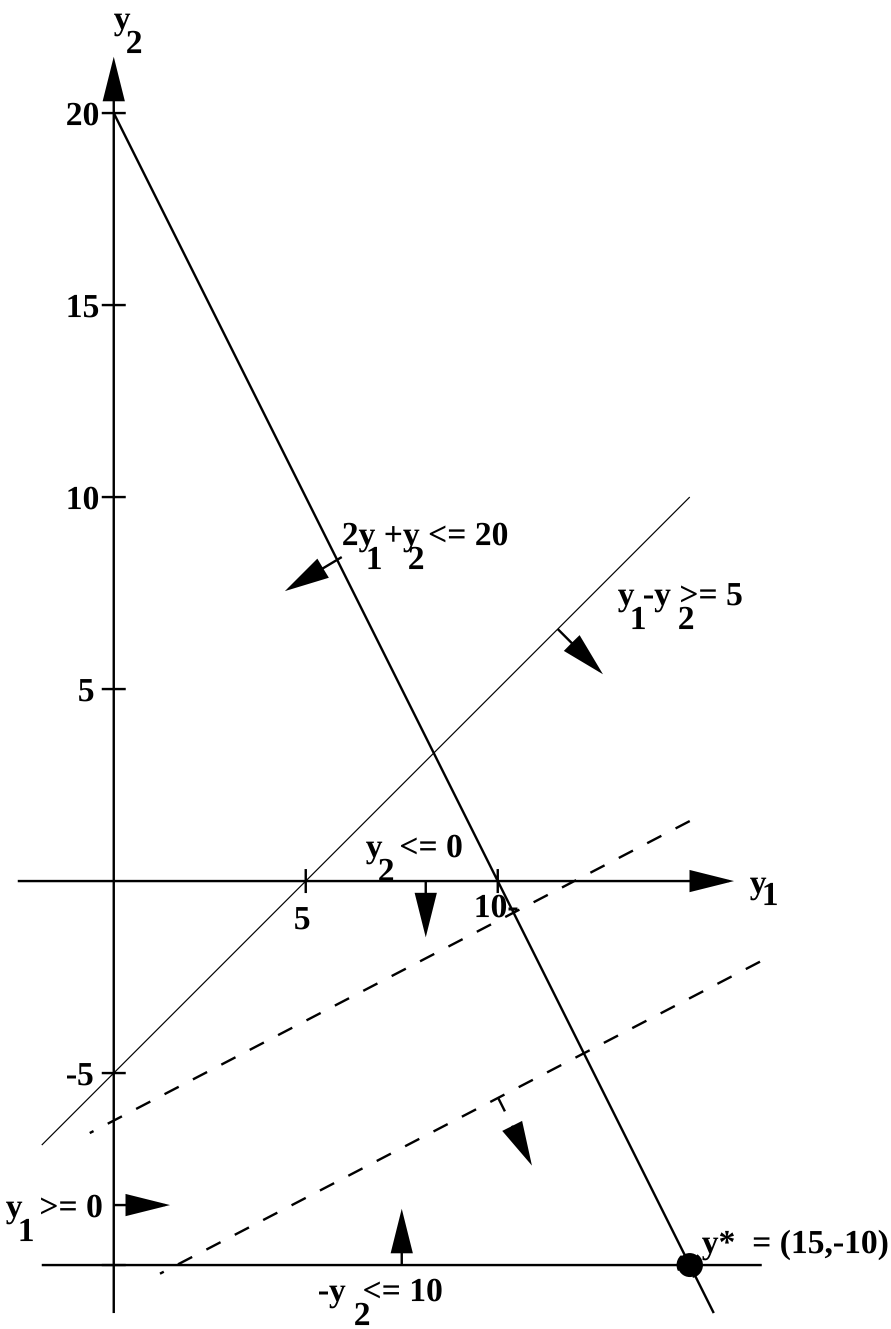
LP model:

$$
\begin{array}{lll}
\text{Maximize} & 100x_1 + 50x_2 + 10(x_3 + 4x_1 + x_2) & (\text{\$ profit}) \\
\\
\text{subject to} & 10x_1 + 10x_2 + 80(x_3 + 4x_1 + x_2) \leq 3600 & (\text{bd.ft. Walnut limit}) \\
& 50x_1 + 30x_2 + 5(x_3 + 4x_1 + x_2) \leq 4300 & (\text{bd.ft. Maple limit}) \\
& 100x_1 + 40x_2 \leq 6550 & (\text{bd.ft. Oak limit}) \\
\\
& x_1, x_2, x_3 \geq 0 &
\end{array}
$$

2. (Total 10%) Solve the following linear program graphically. Be sure to clearly label constraints boundaries and objective function contours as well as the feasible region. Also indicate what solution(s) is(are) optimal.

$$\begin{array}{lrcrcr} \text{Minimize} & -y_1 & + & 2y_2 & & \\ \text{subject to} & y_1 & - & y_2 & \geq & 5 \\ & & & -y_2 & \leq & 10 \\ & 2y_1 & + & y_2 & \leq & 20 \\ & \multicolumn{5}{l}{y_1 \geq 0, y_2 \leq 0} \end{array}$$

Solution:



3. (Total 10%) Write the following problem in LP standard form as a **maximization** problem:

$$\begin{array}{lll} \text{Minimize} & -53x_1 + 33(x_1 - \frac{1}{3}x_3) & \\ \text{subject to} & x_j + 1 \le x_{j+1} & j = 1, 2 \\ & x_1 + x_2 = 10 & \\ & x_1 \le 0, x_3 \ge 0 & \end{array}$$

Simplifying and changing the sense of the objective yields:

$$\text{Maximize } 20x_1 + 11x_3$$

Collecting variables on the left and constants on the right makes the first two constraints:

$$\begin{array}{l} x_1 - x_2 \le -1 \\ x_2 - x_3 \le -1 \end{array}$$

Introducing non-negative slack variables x_4 and x_5 respectively in these two constraints yields:

$$\begin{array}{l} x_1 - x_2 + x_4 = -1 \\ x_2 - x_3 + x_5 = -1 \end{array}$$

Replacing the nonpositive variable x_1 with $-x_6 = x_1$, and replacing unrestricted variable x_2 with the difference $x_2 = x_7 - x_8$ of non-negative variables, yields:

$$\text{Maximize } -20x_6 + 11x_3$$

subject to

$$\begin{array}{rr} -x_6 - x_7 + x_8 + x_4 = & -1 \\ x_7 - x_8 - x_3 + x_5 = & -1 \\ -x_6 + x_7 - x_8 = & 10 \end{array}$$

$$x_3, ..., x_8 \ge 0$$

4. (Total 10%) Consider the following LP standard display:

	x_1	x_2	x_3	x_4	
max c	10	1	0	0	b
	-1	1	4	21	11
	2	6	0	-2	2
	B	N	B	N	

(a) Compute the current basic solution.

Setting nonbasic $x_2 = x_4 = 0$ and solving $Ax = b$ yields the BFS $x = (1, 0, 3, 0)$.

(b) Compute all simplex directions at the current basic solution.

For x_2, we fix $\Delta x_2 = 1, \Delta x_4 = 0$ and solve $A\Delta x = 0$ to obtain $\Delta x = (-3, 1, -1, 0)$.

For x_4, we fix $\Delta x_2 = 0, \Delta x_4 = 1$ and solve $A\Delta x = 0$ to obtain $\Delta x = (1, 0, -5, 1)$.

(c) Verify that the simplex directions at the current basic solution are feasible.

From (b) we have $A\Delta x = 0$ and $\Delta x_j \geq 0$, for all $j \in N$, by construction. Thus each Simplex direction, Δx, is feasible.

(d) Determine whether each of the simplex directions is improving or not.

For x_2, we have $\bar{c} = c \cdot \Delta x = -30 + 1 = -29 < 0$ which tells us that Δx is not improving (for maximization).

For x_4, we have $\bar{c} = c \cdot \Delta x = 10 > 0$ which tells us that Δx is improving (for maximization).

(e) For any improving simplex direction at the current basic solution determine the corresponding maximum feasible step and the new basis that would result after such a step in that direction.

Since $\Delta x = (1, 0, -5, 1)$ is improving we have $\lambda = \min\left(\frac{x_3}{-\Delta x_3}\right) = \frac{3}{5}$ is the maximum feasible step, and the new basis that would result after this step is $B = \{x_1, x_4\}$.

5. (Total 15%) Consider the following LP:

$$\begin{array}{lrcrcrcl} \text{Maximize} & 10x_1 & + & 4x_2 & + & 3x_3 & & \\ \text{subject to} & x_1 & & & - & x_3 & \leq & 0 \\ & -x_1 & + & x_2 & + & 3x_3 & \leq & 4 \\ & x_1 & + & x_2 & & & \leq & 3 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Begin by (a) putting the above problem into LP Standard Form. Then (b) form the initial **dictionary** corresponding to this LP. Next (c) perform, via dictionaries, two complete pivots (i.e. iterations) of the simplex method as described in your handouts (including all the corresponding information). Finally (d) interpret the solution in your final dictionary.

(Note: Please be careful – if you start with an incorrect initial dictionary you can receive only half marks at best!)

(a) Introduce non-negative slacks x_4, x_5, x_6 respectively into the three main constraints to obtain the following LP Standard Form:

$$\begin{array}{lrcrcrcrcrcrcl} \text{Maximize} & 10x_1 & + & 4x_2 & + & 3x_3 & & & & & & & & \\ \text{subject to} & x_1 & & & - & x_3 & + & x_4 & & & & & = & 0 \\ & -x_1 & + & x_2 & + & 3x_3 & & & + & x_5 & & & = & 4 \\ & x_1 & + & x_2 & & & & & & & + & x_6 & = & 3 \end{array}$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

(b) For $B = \{x_4, x_5, x_6\}$ we get the following initial dictionary:

$$\begin{array}{rclcrcrcr} x_4 & = & & - & x_1 & & & + & x_3 \\ x_5 & = & 4 & + & x_1 & - & x_2 & - & 3x_3 \\ x_6 & = & 3 & - & x_1 & - & x_2 & & \\ \\ z & = & & & 10x_1 & + & 4x_2 & + & 3x_3 \end{array}$$

(c) For $EBV = \{x_1\}$, $LBV = \{x_4\}$, and pivot equation: $x_1 = x_3 - x_4$ we get the following dictionary:

$$\begin{array}{rclcrcrcr} x_1 & = & & & & & x_3 & - & x_4 \\ x_5 & = & 4 & - & x_2 & - & 2x_3 & - & x_4 \\ x_6 & = & 3 & - & x_2 & - & x_3 & + & x_4 \\ \\ z & = & & & 4x_2 & + & 13x_3 & - & 10x_4 \end{array}$$

with $B = \{x_1, x_5, x_6\}$.

Part (c) continued.

For $EBV = \{x_3\}$**,** $LBV = \{x_5\}$**, and pivot equation:** $x_3 = 2 - \frac{1}{2}x_2 - \frac{1}{2}x_4 - \frac{1}{2}x_5$ **we get the following dictionary:**

$$
\begin{array}{rcrcrcrcr}
x_1 & = & 2 & - & \frac{1}{2}x_2 & - & \frac{3}{2}x_4 & - & \frac{1}{2}x_5 \\
x_3 & = & 2 & - & \frac{1}{2}x_2 & - & \frac{1}{2}x_4 & - & \frac{1}{2}x_5 \\
x_6 & = & 1 & - & \frac{1}{2}x_2 & + & \frac{3}{2}x_4 & + & \frac{1}{2}x_5 \\
z & = & 26 & - & \frac{5}{2}x_2 & - & \frac{33}{2}x_4 & - & \frac{13}{2}x_5
\end{array}
$$

with $B = \{x_1, x_3, x_6\}$.

(d) This final dictionary corresponds to the BFS $x = (2, 0, 2, 0, 0, 1)$ **which is optimal since the reduced cost** $\bar{c}$ **is negative for every nonbasic variable (**$-\frac{5}{2}$ **for** x_2**,** $-\frac{33}{2}$ **for** x_4**,** $-\frac{13}{2}$ **for** x_5**) and this problem is a maximization LP.**

6. (Total 10%) Solve the following two linear programs:

(a)

$$\begin{array}{lrcrcr} \text{Minimize} & 3x_1 & + & 5x_2 & & \\ \text{subject to} & x_1 & + & x_2 & \geq & 50 \\ & x_1 & + & x_2 & \leq & 95 \\ & & & x_2 & \leq & 30 \\ & x_1 & & & \geq & 0 \end{array}$$

This problem is unbounded and thus has no (finite) solution.

Proof:

To prove this, by example, note that the objective function goes to $-\infty$ **as** x_1 **increases in value, from** $x_1 = 20$**, along the line** $x_1 + x_2 = 50$**. Since the points** $(x_1, x_2) = (20 + \lambda, 30 - \lambda)$ **on this path, are feasible for all** $\lambda \geq 0$ **and yield an objective value of** $210 - 2\lambda$ **the problem is unbounded.**

(b)

$$\begin{array}{lrcrcr} \text{Maximize} & 450v & + & 200c & & \\ \text{subject to} & 10v & + & 7c & \leq & 70000 \\ & v & + & c & = & 10000 \\ & v & & & \leq & 7000 \\ & & & c & \leq & 7000 \end{array}$$

This problem is infeasible and thus has no solution.

Proof:

The equation $v + c = 10000$ **implies that** $7v + 7c = 70000$**.**
For this to hold simultaneously with the constraint $10v + 7c \leq 70000$ **requires** $v = 0$ **and** $c = 10000$**.**
But the last constraint makes this value of c **impossible. Therefore the feasible region is empty.**

Multiple Choice problems, 6@2% + 1@3% = Total 15%

7. In each of the following circle whichever choice provides the best answer.

The first two questions relate to the following standard form linear program Model 1:

$$\begin{array}{lrcrcrcl} \text{Minimize} & 3x_1 & + & 2x_2 & + & 9x_3 & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & = & 9 \\ & x_1 & - & x_2 & & & = & 4 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0 \end{array}$$

(a) Which of the following is an extreme point solution to Model 1?

(i) $x = (3, 6, 0)$ ((iii) $x = (4, 0, 5)$)

(ii) $x = (3, 3, 3)$ (iv) $x = (6, 2, 1)$

(b) Which of the following is **not** required of a feasible direction at $x = (6.5, 2.5, 0)$ in Model 1?

(i) $\Delta x_1 + \Delta x_2 + \Delta x_3 = 0$ ((iii) $\Delta x_1 \geq 0$)

(ii) $\Delta x_1 - \Delta x_2 = 0$ (iv) $\Delta x_3 \geq 0$

The next two questions relate to the following maximizing standard form linear program Model 2:

	w_1	w_2	w_3	w_4	w_5	
max c	-1	3	0	7	-2	b
A	2	-1	1	4	1	8
	0	1	3	0	3	12

(c) If variables w_1 and w_3 are currently basic in Model 2, the simplex direction for w_5 in the next move in Simplex solution of Model 2 is:

(i) $\Delta w = (0, 0, 1, 0, 1)$ (iii) $\Delta w = (2, 0, 4, 0, 1)$

((ii) $\Delta w = (0, 0, -1, 0, 1)$) (iv) $\Delta w = (0, 0, -1, 0, 0)$

(d) The maximum feasible step α for a move in the direction $\Delta w = (-6, -1, 1, 0, 0)$ from solution $w = (6, 6, 1, 0, 1)$ of Model 2 is:

(i) 4 (iii) 2

(ii) 3 ((iv) 1)

The final three Multiple Choice questions are:

(e) If the objective function of a general linearly constrained mathematical program is unimodal then every local optimum is a global optimum.

((i) True) (ii) False

(f) If the main (i.e. functional) constraints of a general mathematical program are linear then its feasible set is necessarily convex.

(i) True ((ii) False)

(g) (3%) Suppose the two-phase improving search of a general mathematical program with original variables x_1, x_2, and x_3 terminates with the following Phase 1 solutions in different cases. Identify all cases that would proceed with Phase II.

(i) Global optimum $x = (40, 7, 0, 9, 0)$. ((iii) Local optimum $x = (1, 3, 1, 0, 0)$)

((ii) Global optimum $x = (22, 4, 3, 0, 0)$) (iv) Local optimum $x = (0, 2, 6, 5, 1)$, which may not be a global optimum.

8. (Total 20%) A refinery takes three raw gasolines and blends them to produce two types of fuel. Here are the data:

Raw gas type	Octane rating (Oc.R)	Supply (barrels/day)	Cost ($/barrel)
1	68	4000	31.02
2	86	5050	33.15
3	91	7100	36.35

Fuel type	Minimum Oc.R	Selling Price ($/barrel)	Demand (barrels/day)
1	95	45.15	At most 10,000
2	85	40.99	At least 15,000

The company sells raw gasoline not used in making fuels at $38.95/barrel if its octane rating is greater than 90, and at $36.85/barrel otherwise.

Formulate an LP model to determine the compositions of the two fuels to maximize total daily profit. Be sure to state any assumptions that you've made!

Assumption

The Oc.R. of each fuel is the weighted average of the Oc.R. of the gases blended to produce each fuel; the weights being the proportions of the constituents.

Parameters:

Let o_i denote the octane rating of fuel type i for $i = 1, 2, 3$.

Variables:

Let $x_{i,j}$ denote the number of barrels per day of raw gas type i used in making fuel type j for $i = 1, 2, 3$ and $j = 1, 2$.

Let y_i denote the number of barrels per day of raw gas type i sold as is, for $i = 1, 2, 3$.

LP Model:

Maximize $45.15 \sum_{i=1}^{3} x_{i,1} + 40.99 \sum_{i=1}^{3} x_{i,2}$ — \$ daily profit from fuel sales -

$-31.02 \sum_{j=1}^{2} x_{1,j} - 33.15 \sum_{j=1}^{2} x_{2,j} - 36.35 \sum_{j=1}^{2} x_{3,j}$ — \$ daily cost of gas constituents +

$+(36.85 - 31.02)y_1 + (36.85 - 33.15)y_2 + (38.95 - 36.35)y_3$ — \$ daily profit from sale of raw gas

subject to

$\sum_{i=1}^{3}(o_i - 95)x_{i,1} \geq 0$ — min Oc.R. for fuel type 1

$\sum_{i=1}^{3}(o_i - 85)x_{i,2} \geq 0$ — min Oc.R. for fuel type 2

$y_1 + \sum_{j=1}^{2} x_{1,j} \leq 4000$ — availability of gas type 1; barrels/day

$y_2 + \sum_{j=1}^{2} x_{2,j} \leq 5050$ — availability of gas type 2; barrels/day

$y_3 + \sum_{j=1}^{2} x_{3,j} \leq 7100$ — availability of gas type 3; barrels/day

$\sum_{i=1}^{3} x_{i,1} \leq 10,000$ — demand for fuel type 1; barrels/day

$\sum_{i=1}^{3} x_{i,2} \geq 15,000$ — demand for fuel type 2; barrels/day

$x_{i,j}, y_i \geq 0$ — $i = 1, 2, 3; j = 1, 2$

Note:

The first two constraints come from applying the assumption. For example, since the total amount of fuel type 1 produced daily is $\sum_{i=1}^{3} x_{i,1}$, by the assumption its Oc.R is $(\sum_{i=1}^{3} o_i x_{i,1})/\sum_{i=1}^{3} x_{i,1}$ which is required to be at least 95. Writing this requirement as a linear inequality yields the first constraint.