SD 252 Quiz 1

$3.78 + TAX

SYDE 252 **QUIZ 1** **Oct. 15, 1992**

1. Suppose a system has step response

$$g(t) = u(t) * h(t) = \begin{cases} \sin 2\pi t & 0 \le t \le 1 \\ 0 & \text{else.} \end{cases}$$

   a. Find and sketch the system's impulse response, $h(t)$.

   b. Find and sketch the output $y(t)$ if the input is

$$x(t) = u(t) - u(t-1).$$

   c. Find the input $x(t)$ which produces a response

$$y(t) = [\sin 2\pi t]u(t).$$

2. In image processing a reasonable model of image blur is a linear shift invariant system with impulse response

$$h(n) = a^{|n|}$$

Suppose $a = 1/2$.

A simple model of lateral inhibition, a neural mechanism in the retina in which neighbouring inputs inhibit the response, is the linear difference equation:

$$y(n) = \frac{5}{3}x(n) - \frac{2}{3}x(n-1) - \frac{2}{3}x(n+1).$$

   a. Find $h_i(n)$, the impulse response of lateral inhibition.

   b. What would be the result if an image blurred by $h(n)$ is processed by lateral inhibition? Explain.

3. Suppose a discrete system has the LDE:

$$y(n) - \frac{10}{3}y(n-1) + y(n-2) = x(n).$$

and is initially at rest.

a. Find the impulse response $h(n)$.

b. Is the system stable? Explain.

c. Find

$$h(n) * [\delta(n) - \frac{10}{3}\delta(n-1) + \delta(n-2)].$$

Is the system invertible? Explain.

4. Suppose a continuous system has the LDE:

$$y''(t) + y'(t) + \frac{17}{4}y(t) = x(t)$$

a. Find the impulse response, $h(t)$.

b. Suppose the system is changed to:

$$y''(t) + y'(t) + \frac{17}{4}y(t) = \frac{1}{2}x(t) + x'(t).$$

What is the impulse response of this system?

SYDE 252 QUIZ 1 SOLUTIONS 15/10/92

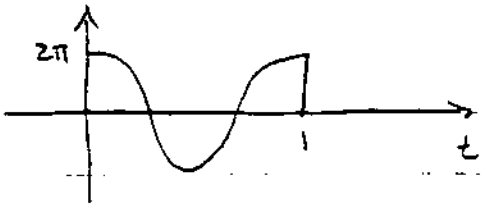
Ed Jernigan

1. a. $g(t) = u(t) * h(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \quad \therefore \; h(t) = g'(t)$

$g(t) = \sin 2\pi t \,[u(t) - u(t-1)]$

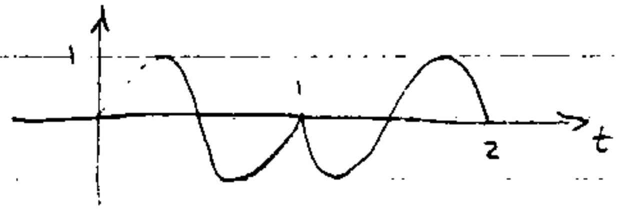
$g'(t) = 2\pi \cos 2\pi t\,[u(t) - u(t-1)] + \underbrace{\sin 2\pi t\,[\delta(t) - \delta(t-1)]}_{0}$

$\therefore \; h(t) = 2\pi \cos 2\pi t\,[u(t) - u(t-1)]$



b. $x(t) = u(t) - u(t-1) \;\Rightarrow\; y(t) = g(t) - g(t-1)$

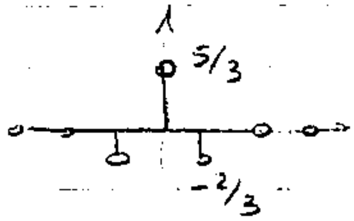
$y(t) = \sin 2\pi t\,[u(t) - u(t-1)] - \sin 2\pi(t-1)\,[u(t-1) - u(t-\ldots$



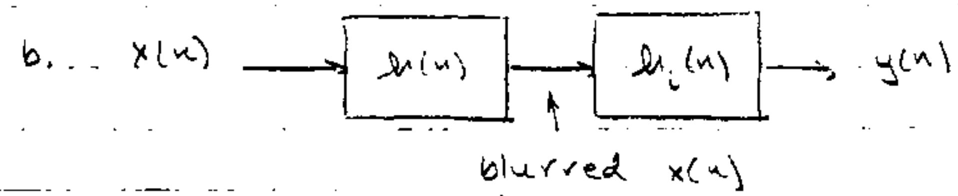
c. For $y(t) = \sin 2\pi t \; u(t) = \sum_{k=0}^{\infty} g(t-k)$

$x(t) = \sum_{k=0}^{\infty} u(t-k)$

2. a. $h_i(n) = \frac{5}{3}\delta(n) - \frac{2}{3}\delta(n-1) - \frac{2}{3}\delta(n+1)$



b.



But $h(n) = (1/2)^{|n|} = \{\ldots \; 1/8 \;\; 1/4 \;\; 1/2 \;\; 1 \;\; 1/2 \;\; 1/4 \;\; 1/8\}$

and $h_i(n) * h(n) = \frac{5}{3}h(n) - \frac{2}{3}h(n-1) - \frac{2}{3}h(n+1)$

3. Suppose a discrete system has the LDE:

$$y(n) - \frac{10}{3}y(n-1) + y(n-2) = x(n).$$

and is initially at rest.

a. Find the impulse response $h(n)$.

b. Is the system stable? Explain.

c. Find

$$h(n) * [\delta(n) - \frac{10}{3}\delta(n-1) + \delta(n-2)].$$

Is the system invertible? Explain.

4. Suppose a continuous system has the LDE:

$$y''(t) + y'(t) + \frac{17}{4}y(t) = x(t)$$

a. Find the impulse response, $h(t)$.

b. Suppose the system is changed to:

$$y''(t) + y'(t) + \frac{17}{4}y(t) = \frac{1}{2}x(t) + x'(t).$$

What is the impulse response of this system?

4\. a. $h''(t) + h'(t) + \frac{17}{4} h(t) = \delta(t)$

$$s^2 + s + 17/4 = 0$$

$$s = \frac{-1 \pm \sqrt{1-17}}{2} = \frac{-1 \pm j4}{2}$$

$$s_1 = -\tfrac{1}{2} + j2 \qquad s_2 = -\tfrac{1}{2} - j2$$

$$h(t) = A e^{(-\frac{1}{2}+j2)t} + B e^{(-\frac{1}{2}-j2)t}$$

$$h(0) = A + B = 0$$

$$h'(0) = (-\tfrac{1}{2} + j2)A + (-\tfrac{1}{2} - j2)B = 1$$

$$A = -B \Rightarrow (-\tfrac{1}{2} + j2)A + (\tfrac{1}{2} + j2)A = 1$$

$$j4A = 1 \qquad A = \tfrac{1}{4j} \qquad B = -\tfrac{1}{4j}$$

$$h(t) = \frac{1}{2} e^{-\frac{1}{2}t}\left[\frac{1}{2j}(e^{j2t} - e^{-j2t})\right] u(t)$$

$$h(t) = \tfrac{1}{2} e^{-\frac{1}{2}t} \sin 2t \; u(t)$$

b. RHS $= \frac{1}{2}x(t) + x'(t)$

$$\therefore h_b(t) = \tfrac{1}{2} h_a(t) + h_a'(t)$$

w/ $h_a'(t) = \left(-\frac{1}{4} e^{-\frac{1}{2}t} \sin 2t + e^{-\frac{1}{2}t} \cos 2t\right) u(t)$

$$+ \tfrac{1}{2} e^{-\frac{1}{2}t} \overbrace{\cancel{\sin 2t\, \delta(t)}}^{\to 0}$$

and $\frac{1}{2} h_a(t) + h_a'(t)$ gives

$$h_b(t) = e^{-\frac{1}{2}t} \cos 2t \; u(t)$$

② $n = 0 \Rightarrow \frac{5}{3} - \frac{2}{3}\cdot\frac{1}{2} - \frac{2}{3}\cdot\frac{1}{2} = \frac{5}{3} - \frac{2}{3} = 1$

③ $n > 0 \quad \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{2}\right)^{n+1} - \frac{2}{3}\left(\frac{1}{2}\right)^{n-1}$

$= \left(\frac{1}{2}\right)^n \left[\frac{5}{3} - \frac{1}{3} - \frac{4}{3}\right] = 0$

$n < 0 \Rightarrow \left(\frac{1}{2}\right)^{-n}\left[\frac{5}{3} - \frac{4}{3} - \frac{1}{3}\right] = 0$

$\therefore h_1(n) * h(n) = \delta(n)$

$\therefore y(n) - x(n)$ [illegible] is the inverse system of the [illegible] model

3. a. $h(n) - \frac{10}{3} h(n-1) + h(n-2) = \delta(n)$

$\Rightarrow 1 - \frac{10}{3} z^{-1} + z^{-2} = 0$

$z^2 - \frac{10}{3} z + 1 = 0$

$(z - 3)(z - \frac{1}{3}) = 0 \qquad z_1 = 3, \; z_2 = \frac{1}{3}$

$h(n) = A\, 3^n + B \left(\frac{1}{3}\right)^n$

$h(-1) = \frac{1}{3} A + 3B = 0$

$h(0) = A + B = 1 \Rightarrow A = 1 - B$

$\Rightarrow \frac{1}{3} - \frac{1}{3} B + 3B = 0$

$\frac{8}{3} B = -\frac{1}{3} \Rightarrow B = -\frac{1}{8} \Rightarrow A = \frac{9}{8}$

$$h(n) = \left[\frac{9}{8}\, 3^n - \frac{1}{8}\left(\frac{1}{3}\right)^n\right] u(n)$$

b. $\sum_n |h(n)| \to \infty$ from 3^n term $\Rightarrow$ unstable

c. $h(n) * \left\{\delta(n) - \frac{10}{3}\delta(n-1) + \delta(n-2)\right\}$

$= h(n) - \frac{10}{3} h(n-1) + h(n-2) = \delta(n)$

from part a.

$\therefore h(n)$ is invertible w/ $h_1(n) = \delta(n) - \frac{10}{3}\delta(n-1) + \delta$

SYDE 252 QUIZ 1 Oct. 13, 1993

1. Suppose a discrete LTI system has a response to a unit step given by:

$$g(n) = \left[\frac{1-a^{n+1}}{1-a}\right]u(n)$$

where $0 < a < 1$.

a. Find the impulse response $h(n)$.

b. Is the system stable? Explain.

c. Show that this system is invertible by finding $h_1(n)$ such that $h(n) * h_1(n) = \delta(n)$.

d. What is the linear difference equation for the system?

2. A system has an impulse response of the form:

$$h(t) = \left(Ae^{-t} + Be^{-2t}\right)u(t).$$

Assume the system's LDE is of the form:

$$y''(t) + a_1 y'(t) + a_0 y(t) = x(t).$$

a. Find a_0 and a_1 to obtain the LDE.

b. Use appropriate initial conditions to find A and B to obtain $h(t)$.

c. Suppose the system is modified to:

$$y''(t) + a_1 y'(t) + a_0 y(t) = x(t) + x'(t).$$

For the same a_0 and a_1, what is $h(t)$ now?

3. A simple model of an echo is a system whose impulse response is:

$$h(t) = \delta(t) + \delta(t - T).$$

Suppose a pure tone of the form:

$$x(t) = \sin 2\pi f t \; u(t)$$

is applied to the system.

a. Find and sketch $y(t)$ if $f = \frac{1}{2T}$.

b. Find and sketch $y(t)$ if $f = \frac{1}{T}$.

c. Suppose you want to cancel the echo with a system whose impulse response is $h_1(t)$. What should $h_1(t)$ be to exactly cancel the echo?

SD 252 Lessons from Quiz 1 19/10/93

Read 186-205 for next time:

Quiz 1 : max 98 min 9 ave 54 !

1. $g(n) = \left[\dfrac{1-a^{n+1}}{1-a}\right] u(n) \qquad 0 < a < 1$

a. $h(n)$: nb. $g(n) = u(n) * h(n) = \sum_{k=-\infty}^{n} h(k)$

note $\sum_{k=0}^{n} a^k = \dfrac{1-a^{n+1}}{1-a}$

$\therefore \quad h(n) = a^n u(n)$

also $\delta(n) = u(n) - u(n-1)$

$h(n) = g(n) - g(n-1)$

$$= \left(\frac{1-a^{n+1}}{1-a}\right)u(n) - \left(\frac{1-a^n}{1-a}\right)u(n-1)$$

$$= \delta(n) + \frac{1-a^{n+1}-1+a^n}{1-a}\, u(n-1)$$

$$= \delta(n) + \frac{1-a}{1-a}\, a^n u(n-1)$$

$$= a^n u(n)$$

b. Stable: $\sum_{k=0}^{\infty} |h(k)| = \sum_{k=0}^{\infty} a^k = \dfrac{1}{1-a} < \infty$

for $0 < a < 1 \quad \therefore$ stable !

c. Find $h_1(n)$ s.t $h(n) * h_1(n) = \delta(n)$

$$h(n) = a^n u(n)$$

$$h(n-1) = a^{n-1} u(n-1)$$

$$\Rightarrow \quad a\,h(n-1) = a^n u(n-1)$$

$$\therefore \quad h(n) - a\,h(n-1) = \delta(n)$$

$$\therefore \quad h(n) * \left[\delta(n) - a\,\delta(n-1)\right] = \delta(n)$$

$$\therefore \quad h_1(n) = \delta(n) - a\,\delta(n-1)$$

① $h_1(n) = 0$, $n<0$ since [illegible] $n<0$

② [illegible] $\sum$ [illegible]

alternately:

$h(k)$	0	0	0	1	a	a^2	a^3
$h_1(n-k)$	0	0	$-a$	1 (n)	0	0	0
$h(k) * h_1(n-k)$	0	0	0	1	0	0	0

d. LDE: $h(n) - a\,h(n-1) = \delta(n)$

$$\therefore \quad y(n) - a\,y(n-1) = x(n)$$

or just refer to our class examples.

2. $h(t) = \left(Ae^{-t} + Be^{-2t}\right)u(t)$

$y''(t) + a_1 y'(t) + a_0 y(t) = x(t)$

a. a_0, a_1 : $s^2 + a_1 s + a_0 = 0$

must have roots $s = -1$, $s = -2$

$\therefore \quad (s+1)(s+2) = 0$

$s^2 + 3s + 2 = 0$

$\Rightarrow \quad a_0 = 2 \; , a_1 = 3$

$y''(t) + 3y'(t) + 2y(t) = x(t)$

b. A & B : $h(0) = 0$, $h'(0) = 1$

$$\Rightarrow \left.\begin{aligned} A + B &= 0 \\ -A - 2B &= 1 \end{aligned}\right\} \quad B = -1 \; , \; A = 1$$

$h(t) = \left(e^{-t} - e^{-2t}\right)u(t)$

c. $h(t) = h_1(t) + h_1'(t)$

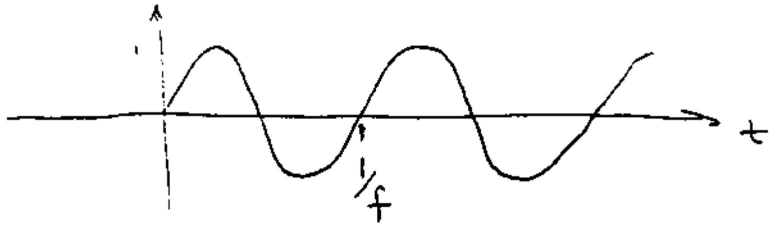
$= \left(e^{-t} - e^{-2t}\right)u(t) + \left(-e^{-t} + 2e^{-2t}\right)u(t)$

$+\left(e^{-t} - e^{-2t}\right)\overset{\to 0}{\delta(t)}$

$h(t) = e^{-2t}\,u(t)$

3. Echo $h(t) = \delta(t) + \delta(t-T)$

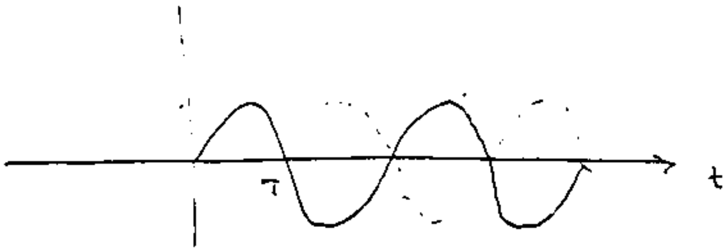
$x(t) = \sin 2\pi f t \;\; u(t)$



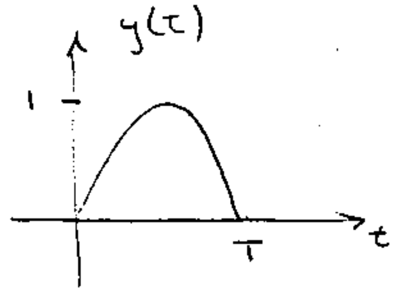
a. $y(t) = x(t) + x(t-T)$ w/ $f = 1/2T$

$$= \sin \frac{\pi t}{T}\, u(t) + \sin \frac{\pi(t-T)}{T} \; u(t-T)$$

$$= \begin{cases} \sin \frac{\pi t}{T} & 0 \le t \le T \\ 0 & \text{else} \end{cases}$$

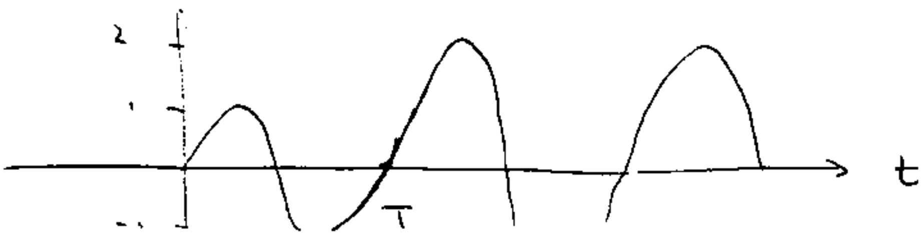


$\Rightarrow$



b. $y(t) = \sin \frac{2\pi t}{T}\; u(t) + \sin \frac{2\pi(t-T)}{T}\; u(t-T)$

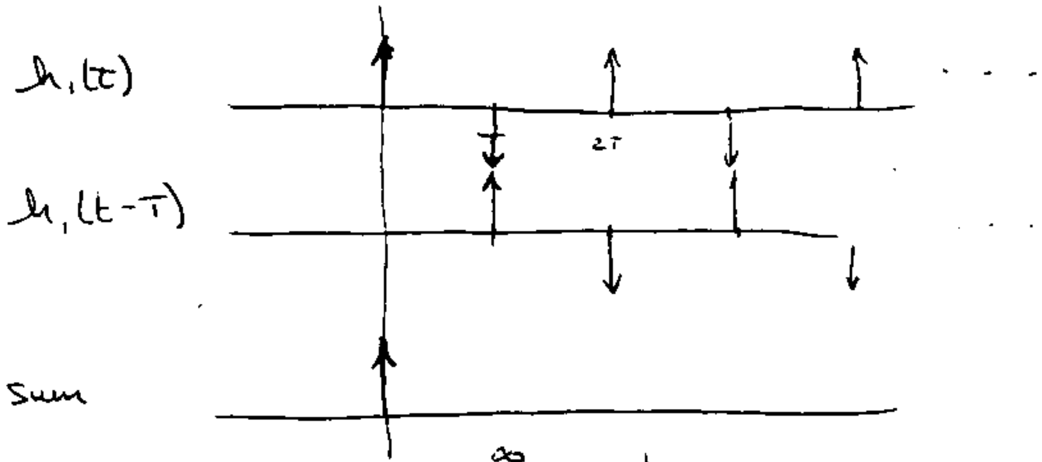
$$= \begin{cases} \sin \frac{2\pi t}{T} & 0 \le t \le T \\ 2 \sin \frac{2\pi t}{T} & t > T \end{cases}$$



c. $h_1(t)$ s.t. $h(t) * h_1(t) = \delta(t)$

$$\Rightarrow h_1(t) * \left[\delta(t) + \delta(t-T)\right] = \delta(t)$$

$$\Rightarrow h_1(t) + h_1(t-T) = \delta(t)$$



$$\therefore \quad h_1(t) = \sum_{k=0}^{\infty} (-1)^k \delta(t - kT)$$

SYDE 252 QUIZ 1 Oct. 13, 1994

Notes:

1. All parts worth 10 marks each.

2. If you need a specific value of T or a in problem 1 to get started, try $T=2$ and $a=1$. You'll need the general form for full marks.

3. $x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$; $x(n)*h(n)=\sum_{k=-\infty}^{\infty}x(k)h(n-k)$.

1. **Evaluate** and carefully **sketch** the following signals:

a. $u(t-T)*u(t+T)$.

b. $\left(\frac{1}{2}\right)^{|n|}*[-2\delta(n+1)+5\delta(n)-2\delta(n-1)]$.

c. $e^{-at}u(t)*[u(t)-u(t-T)]$.

2. A discrete LTI system has linear difference equation:

$$y(n)-1.7y(n-1)+.72y(n-2)=x(n).$$

a. Find the **impulse response**, $h(n)$.
b. Is the system **stable? Prove or disprove.**
c. Is the system **invertible?** Find the **inverse system** if so.
d. What is the **impulse response** if the right hand side of the equation is $x(n)-.9x(n-1)$?

3. A second order LTI system has linear differential equation:

$$y''(t)+2y'(t)+5y(t)=x(t)$$ ← $\delta(t)$. find impulse resp.

a. Find the **impulse response,** $h(t)$.
b. Draw the **Direct Form II block diagram.**
c. If the input is $x(t)=\delta'(t)+2\delta(t)+5u(t)$, what is the **output?**

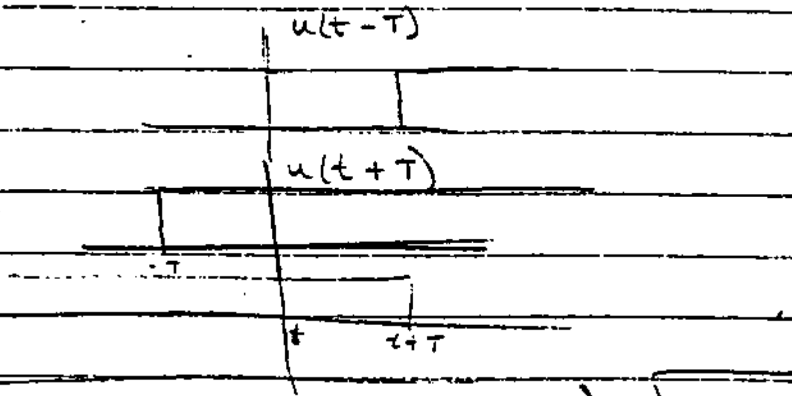
∴ output is . $y(t)=x(t)*h(t)$ $=5\int_{-\infty} h(\tau)d\tau$

$y(t)=h'(t)+2h(t)+5\int_{-\infty}^{t}h(\tau)d\tau$

ED 252 QUIZ 1 Review Oct 17 95

Quiz 1 1994:

1 a. $u(t-T) * u(t+T)$



$$\left.\begin{array}{ll} t < 0 & \Rightarrow 0 \\ t > 0 & \Rightarrow \int_T^{t+T} dt = t+T-T = t \end{array}\right\} \boxed{t\,u(t)}$$

b. $\left(\frac{1}{2}\right)^{|n|} * \underset{-1}{\uparrow}[-2 \quad \underset{0}{\uparrow}5 \quad \underset{1}{\uparrow}-2]$ $\quad \frac{1}{8} \quad \frac{1}{4} \quad \frac{1}{2} \quad 1 \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8}$

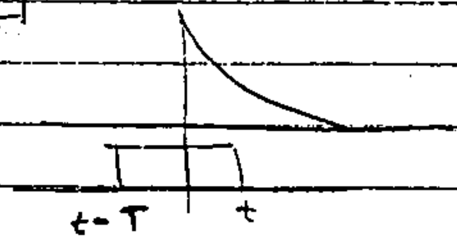
$-2 \quad 5 \quad -2$

$n = 0 \Rightarrow 5 - 1 - 1 = 3$

$n \neq 0 \quad \frac{1}{2}^n 5 - 2\left(\frac{1}{2}\right)^{n-1} - 2\left(\frac{1}{2}\right)^{n+1} = \left(\frac{1}{2}\right)^n \left[5 - 4 - 1\right] = 0$

$\Rightarrow \boxed{3\,\delta(n)}$

c. $e^{-at}u(t) * \left(u(t) - u(t-T)\right)$



$$\left\{\begin{array}{ll} = 0 & t < 0 \\ = \int_0^t e^{-a\tau}d\tau = \frac{1-e^{-at}}{a} & 0 \le t \le T \\ = \int_{t-T}^{t} e^{-a\tau}d\tau = \frac{e^{-a(t-T)} - e^{-at}}{a} & \end{array}\right.$$

$$= e^{-at}\left(\frac{e^{aT}-1}{a}\right) \quad t > T$$

2. $y(n) - 1.7y(n-1) + .72y(n-2) = x(n)$

a $h(n)$: $1 - 1.7z^{-1} + .72z^{-2} = 0 \Rightarrow z^2 - 1.7z + .72 = 0$

$(z-.9)(z-.8) = 0$

$h(n) = A(.9)^n + B(.8)^n$

$h(-1) = \frac{A}{.9} + \frac{B}{.8} = 0 \quad B = -\frac{.8}{.9}A = -\frac{8}{9}A$

$h(0) = A + B = 1 \quad A - \frac{8}{9}A = 1 \quad A = 9$

$B = -8$

$h(n) = \left[9(.9)^n - 8(.8)^n\right]u(n) \qquad \frac{9}{.9} - \frac{8}{.8}$

b. Stable: $\sum|h(n)| \le 9\sum(.9)^n + 8\sum(.8)^n$

$9\frac{1}{1-.9} + 8\frac{1}{1-.8} = 90 + 40 = 130 < \infty$

c Invertible. $h_1(n) = \delta(n) - 1.7\delta(n-1) + .72\delta(n-2)$

$h(n) * h_1(n) = h(n) - 1.7h(n-1) + .72h(n-2) = \delta(n)$

d. RHS: $x(n) - .9x(n-1)$

$h(n) = h_o(n) - .9h_o(n-1) = (.8)^n u(n)$

$\left[9(.9)^n - 8(.8)^n\right]u(n) - .9\left[9(.9)^{n-1} - 8(.8)^{n-1}\right]u(n-1)$

$\delta(n) + (9-.9)\,.9^n - \left(8 - \frac{.9}{.8}8\right)(.8)^n \; u(n-1)$

$= (.8)^n u(n)$

3. $y''(t) + 2y'(t) + 5y(t) = x(t)$

$s^2 + 2s + 5 = 0 \qquad s = \dfrac{-2 \pm \sqrt{4-5}}{2} = -1 \pm \tfrac{1}{2}j$

$h(t) = A e^{-(1-\frac{1}{2}j)t} + B e^{-(1+\frac{1}{2}j)t} \Big|_{t=0} = 0 \qquad B = -A$

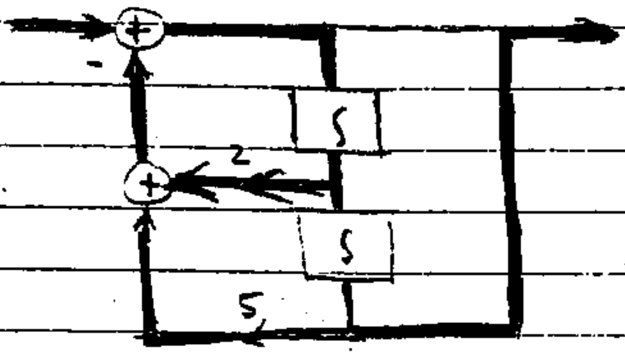
$h'(0) = -(1-\tfrac{1}{2}j)A - (1+\tfrac{1}{2}j)B = 1$

$-(1-\tfrac{1}{2}j)A + (1+\tfrac{1}{2}j)A = 1$

$jA = 1 \qquad A = 1/j \quad B = -1/j$

a. $h(t) = 2e^{-t} \sin \tfrac{1}{2}t \; u(t)$

b. DF II



c. $x(t) = \delta'(t) + 2\delta(t) + 5u(t)$

$x(t) * h(t) = y(t) = h'(t) + 2h(t) + 5\int_{-\infty}^{t} h(\tau)\,d\tau$

but $y'(t) = h''(t) + 2h'(t) + 5h(t) = \delta(t)$

$\therefore \; y(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = \underline{\underline{u(t)}}$

$y(t) = \int x(\tau)\, h(t-\tau)\,d\tau$

83^9 +TAK

SYDE 252 QUIZ 1 Oct. 18, 1995

Aids permitted: 1 $8\frac{1}{2}$x11 page of notes, no calculators.
Each part of each problem is worth 10 marks.

1. Consider the cascade of two causal LTI systems with impulse responses:

$$h_1(n) = a^n u(n) \qquad \text{and} \qquad h_2(n) = b^n u(n).$$

a. Find $h(n)$ in closed form for the complete system for both $a = b$ and $a \neq b$.
b. Use the linear difference equations for the two subsystems to show that the complete system has the LDE: $y(n) - (a+b)y(n-1) + aby(n-2) = x(n)$.
c. Draw the Direct Form II diagram of the system.

2. Suppose a causal LTI system has the linear differential equation:

$$y''(t) + y(t) = x(t).$$

a. Find the impulse response $h(t)$.
b. Find and sketch $y(t)$ if $x(t) = u(t) - u(t-2\pi)$.
c. Is the system stable? Explain. Can you modify the right side of the LDE to achieve a stable system?
d. Find $h(t)$ if the right side of the LDE is $x'(t)$. What if it is $x''(t)$?

3. Let $x(t) = \begin{cases} 1 & n \le t \le n + \frac{1}{2} \\ 0 & \text{else} \end{cases}$ for all integers n.

a. Show that the Fourier Series coefficients are of the form $x_k = \begin{cases} c & k = 0 \\ \dfrac{1}{jk\theta} & k \text{ odd} \\ 0 & \text{else} \end{cases}$.

Find the parameters c and θ.
b. Find the coefficients for the signal $x'(t)$ in terms of the coefficients of $x(t)$.
c. If $x(t)$ is applied to a system for which:

$$H(k2\pi) = \int h(t) e^{-jk2\pi t} dt = \begin{cases} 1 & |k| \le 1 \\ 0 & \text{else} \end{cases},$$

find the output $y(t)$.

1 a. $h(n) = h_1(n) * h_2(n) = \sum_{k=0}^{n} a^k b^{n-k}$ $\Big\}$ $h(n) = \begin{cases} \dfrac{b^{n+1} - a^{n+1}}{b-a} & a \neq b \\ (n+1)a^n & a = b \end{cases}$

b. $x_1(n) \longrightarrow \boxed{h_1(n)} \longrightarrow y_1(n) \longrightarrow x_2(n) \longrightarrow \boxed{h_2(n)} \longrightarrow y_2(n)$

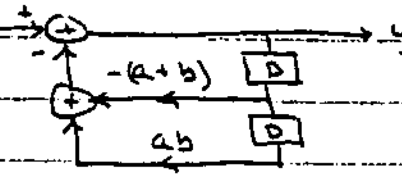
$y_1(n) - a y_1(n-1) = x_1(n)$ $\qquad$ $y_2(n) - b y_2(n-1) = x_2(n)$

Using $y_1(n) = x_2(n)$, $x_1(n) = x(n)$, $y_2(n) = y(n)$:

$y(n) - b y(n-1) - a[y(n-1) - b y(n-2)] = x(n)$

$\Rightarrow y(n) - (a+b) y(n-1) + ab\, y(n-2) = x(n)$ $\qquad$ QED.

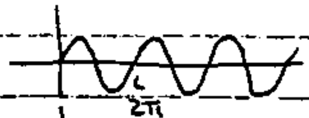
c. DF II: $x(n)$ $y(n)$



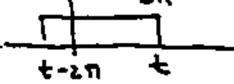
2 a. $h''(t) + h(t) = \delta(t) \Rightarrow s^2 + 1 = 0 \Rightarrow h(t) = Ae^{jt} + Be^{-jt}$

$h'(0) = 1 = jA - jB$; $h(0) = 0 = A + B$ $\quad A = \frac{1}{2j}$ $B = -\frac{1}{2j}$ $\Rightarrow$ $h(t) = \sin t \; u($

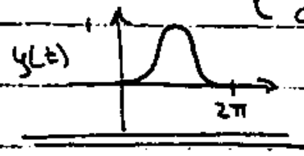
b. ① $h(\tau)$:



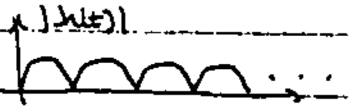
② $x(t-\tau)$



$\Rightarrow y(t) = \begin{cases} 0 & t < 0 \\ \int_0^t \sin\tau \, d\tau = 1 - \cos t & 0 \le t \le \\ 0 & t \geq 2\pi \end{cases}$ (integral over 1 period)



c. Unstable since $\int |h(t)| dt \to \infty$ nb



Stable if RHS is $x''(t) + x(t) \Rightarrow$ then $h(t) = \delta(t)$!

d. $x'(t)$: $h(t) = \frac{d}{dt} \sin t\, u(t) = \cos t\, u(t) + \cancel{\sin t}\,\delta(t) = \cos t\, u(t$

$x''(t)$: $h(t) = \frac{d}{dt} \cos t\, u(t) = \cancel{\cos t}\,\delta(t) - \sin t\, u(t) = \delta(t) - \sin t\, u$

3. a. $X_k = \frac{1}{T}\int_0^T x(t) e^{-jk\omega_0 t} dt = \int_0^{1/2} e^{-jk2\pi t} dt = \dfrac{1 - e^{-jk\pi}}{jk2\pi} = \begin{cases} 0 & k \text{ even}, \neq \\ \frac{1}{jk\pi} & k \text{ odd} \end{cases}$

& $X_0 = \int_0^{1/2} 1\, dt = 1/2$ $\quad \therefore$ $c = 1/2$, $\theta = \pi$

b. $x(t) = \sum_k X_k e^{jk\omega_0 t}$ $\therefore$ $x'(t) = \sum_k jk2\pi X_k e^{jk\omega_0 t}$ $\therefore (X_k)' = jk2\pi X_k$

$\Rightarrow$ coef. of $x'(t) = \begin{cases} 0 & k \text{ even} \\ 2 & k \text{ odd} \end{cases}$

c. $\sum_k X_k e^{jk\omega_0 t} \longrightarrow \boxed{h(t)} \longrightarrow \sum_k X_k H(k\omega_0) e^{jk\omega_0 t}$

$\therefore Y_k = \begin{cases} X_k & k = 0, \pm 1 \\ 0 & \text{else} \end{cases}$ $\quad \therefore$ $y(t) = \frac{1}{2} + \frac{1}{\pi j} e^{j2\pi t} - \frac{1}{\pi j} e^{-j2\pi t}$

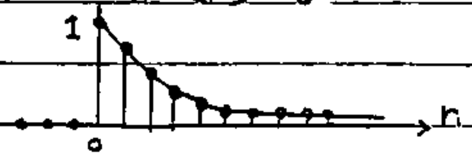
$y(t) = \frac{1}{2} + \frac{2}{\pi} \sin 2\pi t$.

SYDE 252 QUIZ 1 SOLUTIONS — OCT 18 199

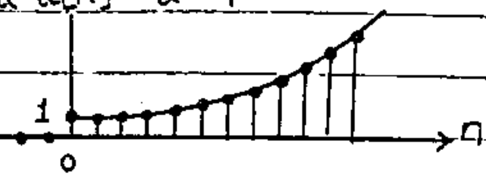
1. Two causal LTI systems are cascaded:

$h_1[n] = a^n u[n] \qquad h_2[n] = b^n u[n]$

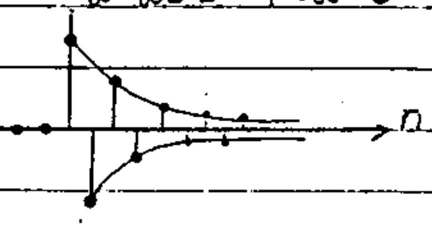
$a^n u[n] \quad 0 < a < 1$



$a^n u[n] \quad a > 1$

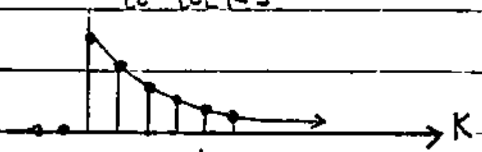


$a^n u[n] \quad -1 < a < 0$

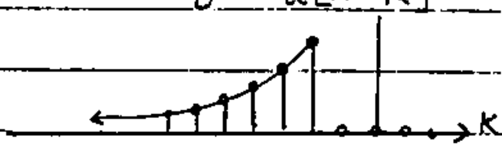


a) Find $h[n] = h_1[n] * h_2[n]$

$a^k u[k]$



$b^{(n-k)} u[n-k]$



Case I: $a \neq b$

$h[n] = 0 \quad ; \; n < 0$

$$h[n] = \sum_{k=0}^{n} a^k b^{n-k} = \sum_{k=0}^{n} a^k b^{-k} b^n = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k = b^n \frac{\left(1 - \left(\frac{a}{b}\right)^{n+1}\right)}{1 - \frac{a}{b}} \quad ; n \geq 0$$

$$\therefore h[n] = \left[\frac{b^{n+1} - a^{n+1}}{b - a}\right] u[n] \qquad a \neq b$$

Case II: $a = b$

$h[n] = 0 \quad ; \; n < 0$

$$h[n] = \sum_{k=0}^{n} a^k a^{n-k} = a^n \sum_{k=0}^{n} 1 = (n+1) a^n \quad ; \; n \geq 0$$

$$\therefore h[n] = (n+1) a^n u[n]$$

1b) Linear difference equations.

We need to find some manipulation of the impulse response's terms which yields a value of 1 at $n=0$, and a value of 0 everywhere else.

Notice: $h_1[n] = a h_1[n-1]$; $n \neq 0$

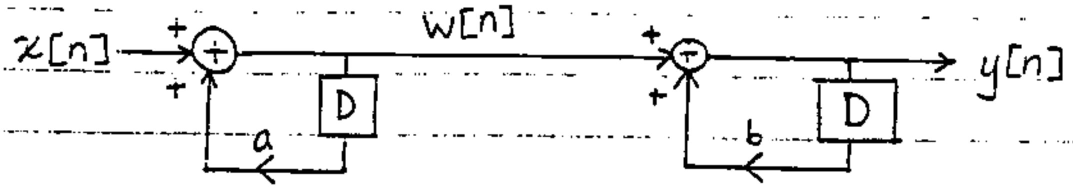
$\therefore h_1[n] - a h_1[n-1] = 0$; $n \neq 0$

$h_1[n] - a h_1[n-1] = 1$; $n = 0$

The LDE's are:

$y_1[n] - a y_1[n-1] = x[n]$

$y_2[n] - b y_2[n-1] = x[n]$



$w[n] = x[n] + a w[n-1] \rightarrow w[n] - a w[n-1] = x[n]$ ①

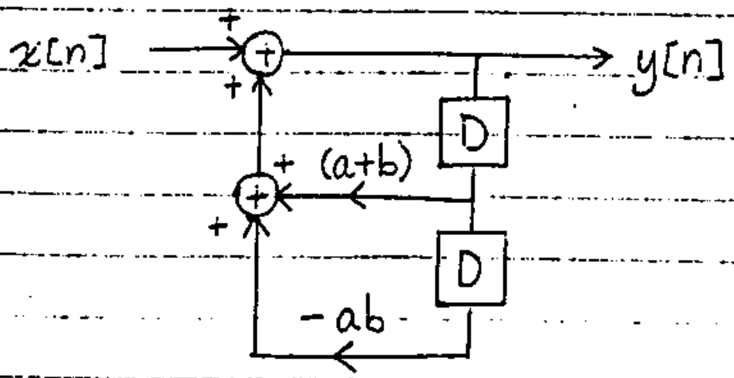
$y[n] = w[n] + b y[n-1] \rightarrow w[n] = y[n] - b y[n-1]$ ②

Sub. ② into ① to eliminate $w[n]$

$y[n] - b y[n-1] - a y[n-1] + ab y[n-2] = x[n]$

$\therefore$ The LDE is: $y[n] - (a+b) y[n-1] + ab y[n-2] = x[n]$

1c) $y[n] = x[n] + (a+b) y[n-1] - ab y[n-2]$



2. Suppose a causal LTI system has the linear differential equation

$$y''(t) + y(t) = x(t) \qquad (1)$$

a. Find the impulse response $h(t)$.

In eqn (1), Let $y(t) = h(t)$ and $x(t) = \delta(t)$:

$$h''(t) + h(t) = \delta(t). \qquad (2)$$

Since the system is causal, we know $h(0^-) = 0$, $h'(0^-) = 0$, $h''(0^-) = 0$.

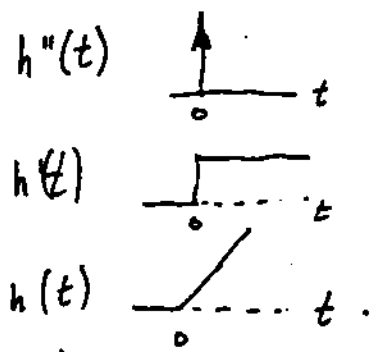
Integrating (2) from 0^- to 0^+,

$$\int_{0^-}^{0^+} h''(t)\,dt + \int_{0^-}^{0^+} h(t)\,dt = \int_{0^-}^{0^+} \delta(t)$$

So $h'(0^+) - h'(0^-) + \quad = 1.$

We know $h'(0^-) = 0$, so $\therefore$ $h'(0^+) = 1.$

Since $h''(t)$ contains an impulse, $h'(t)$ is discontinuous and $h(t)$ is continuous:



Hence $h(0^+) = h(0^-) = 0.$

2a. cont'd.

So our initial conditions are

$$h(0^+) = 0, \quad h'(0^+) = 1.$$

Now we can solve the homogeneous form

$$h''(t) + h(t) = 0$$

to recover the impulse response for $t > 0$.

Characteristic eqn: $\lambda^2 + 1 = 0$

$$\lambda = \pm j.$$

So $h(t) = Ae^{jt} + Be^{-jt}$.

$$h(0^+) = A + B = 0.$$

$$h'(0^+) = jA - jB = 1.$$

$\therefore \; A = \frac{1}{2j}, \quad B = -\frac{1}{2j}.$

$$h(t) = \left(\frac{1}{2j} e^{jt} - \frac{1}{2j} e^{-jt}\right) u(t)$$

$$= \sin t \; u(t).$$

b. Find and sketch $y(t)$ if $x(t) = u(t) - u(t-2\pi)$.

First consider the step response

$$y_{step}(t) = \int_0^t \sin t \, dt = -\cos t \Big|_0^t = (1 - \cos t)u$$

The response to $u(t) - u(t-2\pi)$ is simply

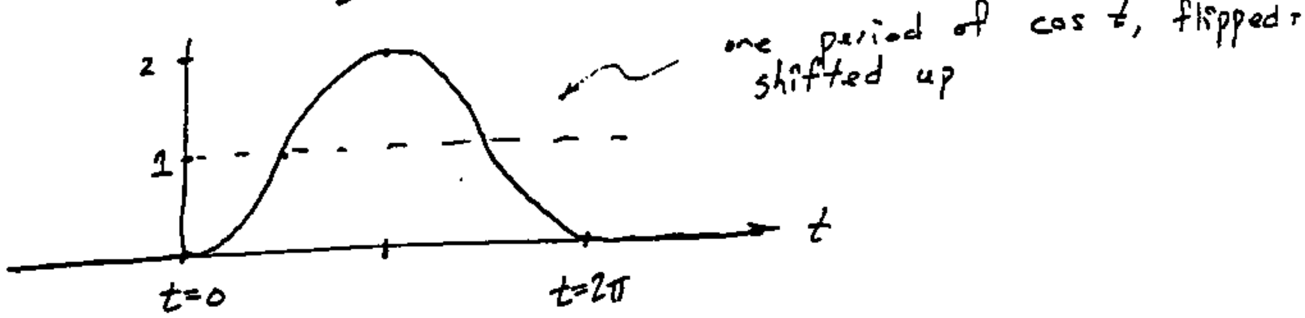
$$y(t) = y_{step}(t) - y_{step}(t-2\pi)$$

$$= (1-\cos t)u(t) - (1 - \cos(t-2\pi))u(t-2\pi)$$

2b cont'd.

Since $\cos(t-2\pi) = \cos t$,

$$y(t) = (1 - \cos t)\left[u(t) - u(t-2\pi)\right].$$



c. The system is not stable, since

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{0}^{+\infty} |\sin t|\, dt \to \infty.$$

Can the RHS be modified to create a stable system? Yes. For example, consider

$$y''(t) + y(t) = x''(t) + x(t).$$

Then $y(t) = x(t)$, the impulse response is simply $\delta(t)$ which is a stable system.

d. If we change the RHS to $x'(t)$, this is equivalent to

$$x(t) \to \boxed{\frac{d}{dt}} \xrightarrow{x'(t)} \boxed{h(t)} \xrightarrow{y(t)}$$

The overall impulse response is just $h'(t)$.

product for deriv

$$h'(t) = \frac{d}{dt} h(t) = \frac{d}{dt} \sin t\, u(t) = \cos t\, u(t) + \sin t\, \delta(t)$$

$$= \cos t\, u(t) + 0\, \delta(t)$$

$$= \cos t\, u(t).$$

2a. For $x''(t)$, the impulse response is

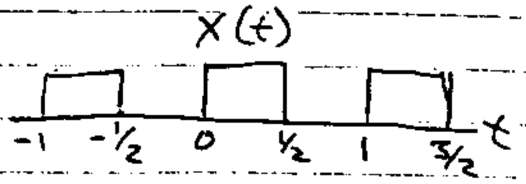
$$\frac{d^2}{dt^2} h(t) = \frac{d}{dt} h'(t)$$

$$= \frac{d}{dt} \cos t \, u(t)$$

$$= -\sin t \, u(t) + \cos t \, \delta(t)$$

$$= -\sin t \, u(t) + \delta(t).$$

3. $x(t) = \begin{cases} 1 & n \le t \le n + \frac{1}{2} \\ 0 & \text{else} \end{cases}, \quad n \in I$



a) $X_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$ $\qquad T = 1, \ \omega_0 = 2\pi$

$= \frac{-1}{2jk\pi} e^{-jk2\pi t} \Big|_0^{1/2}$

$= \frac{1}{2jk\pi} \left[1 - e^{-jk\pi} \right]$

$= \frac{1 - (-1)^k}{2jk\pi} \quad k \neq 0 \quad , \qquad X_0 = \int_0^{1/2} e^{-j\,0\,2\pi t} dt = 1/2$

b) $x'(t) = \frac{d}{dt} \left[X_k e^{+jk\omega_0 t} \right]$

$= +jk2\pi X_k e^{+jk2\pi t}$

$X'_k = +jk2\pi X_k$

$= +1 - (-1)^k$

c) $H_k = \int h(t) e^{-jk2\pi t} dt = \begin{cases} 1 & |k| < 1 \\ 0 & \text{else} \end{cases}$

$y_k = H_k X_k = \begin{cases} -\frac{1}{j\pi} & k = -1 \\ 1/2 & k = 0 \\ 1/j\pi & k = 1 \\ 0 & \text{else} \end{cases}$

$y(t) = 1/2 + \frac{1}{j\pi} \left(e^{j2\pi t} - e^{-j2\pi t} \right)$

$= 1/2 + \frac{2 \sin 2\pi t}{\pi}$

SYDE 252 QUIZ 1 Oct. 9, 1996

Aids permitted: 1 $8\frac{1}{2}$x11 page of notes, no calculators.
Each part of each problem is worth 10 marks.

1. For each of the following systems, find and sketch the output for the given input.

   a. $h(t) = u(t) - u(t-2)$; $\qquad x(t) = e^{-(t-1)}u(t-1)$.

   b. $h(n) = \delta(n) - a\delta(n-1)$; $\qquad x(n) = \left[\dfrac{1-a^{n+1}}{1-a}\right]u(n)$.

2. Suppose two linear time invariant systems have the same response to one particular input. Are the systems identical? Explain. How does the input affect your answer?

3. A simple model of a light meter can be inferred from its response to a unit step representing a unit discontinuity in the input light level. Suppose this step response is:

$$g(t) = \left[2 - 2e^{-t/2}\right]u(t).$$

   a. Find the impulse response of the light meter.
   b. Show that the meter can be represented by the LDE $y'(t) + ay(t) = x(t)$. Find a.
   c. Devise a system which recovers the input to the light meter from its output. Give its LDE and draw its block diagram using integrators.

4. A causal discrete oscillator has the linear difference equation:

$$y(n) - \sqrt{2}y(n-1) + y(n-2) = x(n) - \frac{\sqrt{2}}{2}x(n-1).$$

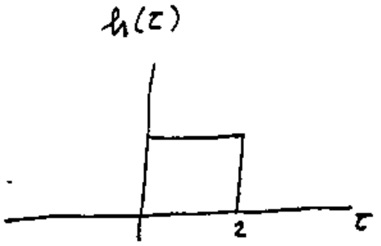
   a. Draw the Direct Form II block diagram.
   b. Find the impulse response by recursively solving the LDE directly for $h(0)$, $h(1)$, $h(2)$, $\cdots$, for at least one complete period.
   c. Find the impulse response of $y(n) - \sqrt{2}y(n-1) + y(n-2) = x(n)$.
   d. Use linearity and time invariance to find the impulse response of the entire system.

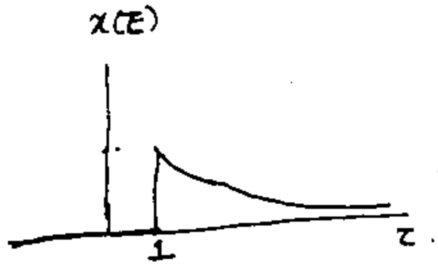
SD 252
OCT 9, 1996
MIDTERM #1
SOLUTIONS.
#1, 2 Srikanth
#3 Janine
#4 Todd.

1. Given $h(t)$ & $x(t)$ find and sketch output.

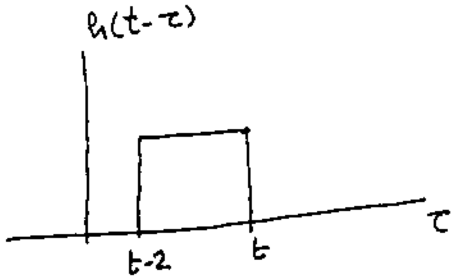
$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau.$$

a. $h(t) = u(t) - u(t-2)$. $\qquad x(t) = e^{-(t-1)}u(t-1)$





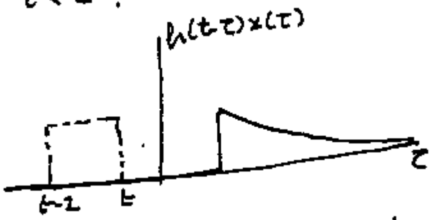
Flip & Shift $h(\tau)$ to get $h(t-\tau)$.



When we move $h(t-\tau)$ for values of t from $-\infty$ to ∞, we see three distinct intervals.

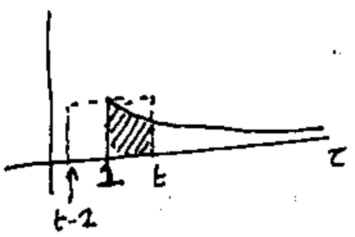
(i) $t<1$. In this case the product $h(t-\tau)x(\tau)$ is zero.

$x(\tau)$ is zero for all values of $t<1$.



$\therefore\ y(t) = 0$ for $t<1$ —— (I)

(ii) $t>1$, but $t-2<1$ i.e. $t<3$. the interval is $1<t<3$.



$y(t)$ is the area of the shaded part

$$\therefore\ y(t) = \int_1^t e^{-(t-1)}dt = \left[-e^{-(t-1)}\right]_1^t$$

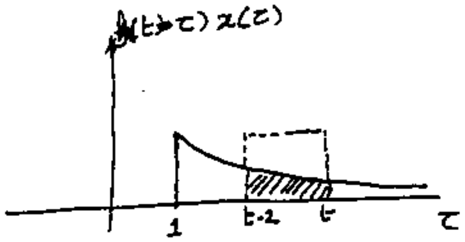
$$= -e^{-(t-1)} - \left(-e^{-(1-1)}\right)$$

$$= -e^{-(t-1)} + 1.$$

Note: $\int e^{at+b}dt = \frac{1}{a}e^{at+b}$

$y(t) = 1 - e^{-(t-1)}$ for $1 \le t < 3$ $\Rightarrow$ (II)

(iii) $t-2 \geq 1$. ie $t \geq 3$



$y(t)$ is the area of the shaded region.

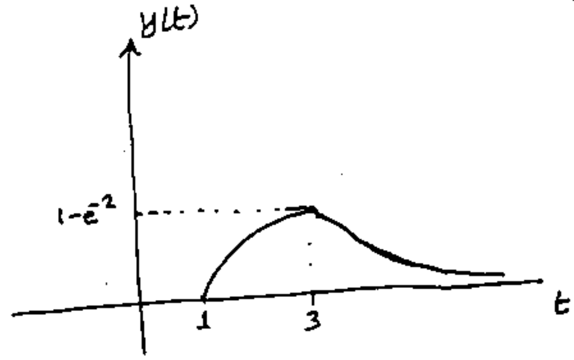
$$y(t) = \int_{t-2}^{t} e^{-(t-1)} dt = \left[-e^{-(t-1)} \right]_{t-2}^{t}$$

$$= -e^{-(t-1)} + e^{-(t-2-1)}$$

$$= e^{-(t-3)} - e^{-(t-1)}$$

$$y(t) = e^{-(t-3)} - e^{-(t-1)} \quad \text{for } t \geq 3 \quad \text{III}$$

from I, II & III
$$y(t) = \begin{cases} 0 & t < 1 \\ 1 - e^{-(t-1)} & 1 < t < 3 \\ e^{-(t-3)} - e^{-(t-1)} & t \geq 3 \end{cases}$$



(b). $h[n] = \delta[n] - a\delta[n-1]$ $\qquad$ $x(n) = \left[\frac{1-a^{n+1}}{1-a} \right] u[n]$.

{The method used in 1.a can be used here to solve for $y[n]$; an alternate method is shown below}.

Observe that the expression $\frac{1-a^{n+1}}{1-a}$ is the closed form of $\sum_{k=0}^{n} a^k$.

ie $\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ $\qquad$ Therefore $x[n] = \sum_{k=0}^{n} a^k u[n]$.

$y[n] = x[n] * h[n] = x[n] * (\delta[n] - a\delta[n-1])$ {use linearity}

$= x[n] * \delta[n] - x[n] * a\delta[n-1]$.

~~Multiplying~~ convolving "anything" with $\delta(n)$ gives "anything". ~~Multiplying~~ Convolving $x[n]$ with $\delta[n-1]$ shifts $x[n]$ by 1.

$$y[n] = x[n] * \delta[n] - a\, x[n] * \delta[n-1]$$
$$= x[n] - a\, x[n-1].$$

Substituting for $x[n]$.

$$y[n] = \sum_{k=0}^{n} a^k u[n] - a \sum_{k=0}^{n-1} a^k u[n-1]$$

$$= \sum_{k=0}^{n} a^k u[n] - \sum_{k=1}^{n} a^k u[n-1]$$

Note that the range of summation changes to $1 \to N$ since
$a(1 + a + a^2) = a + a^2 + a^3$
$a \sum_{k=0}^{2} a^k = \sum_{k=1}^{3} a^k$.

$$= a^0 u[n] + \sum_{k=1}^{n} a^k u[n] - \sum_{k=1}^{n} a^k u[n-1]$$

$$= u[n] + \sum_{k=1}^{n} a^k \left(u[n] - u[n-1]\right)$$

Note: $u[n] - u[n-1] = \delta[n]$

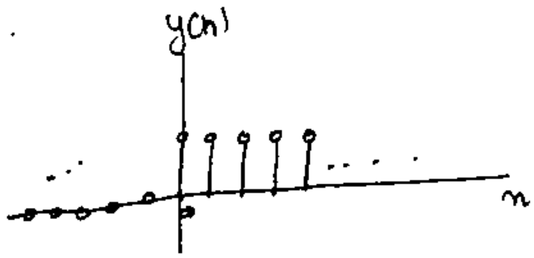
$$= u[n] + \underbrace{\sum_{k=1}^{n} a^k \delta[n]}_{=0}.$$

since $\delta[n] = 1$ for $n=0$ & ϕ otherwise but in the summation there no term for $n=0$.

$$= u[n]$$

$$\therefore \quad \underline{y[n] = u[n]}$$

Sketch



2) If two linear time invariant systems have the same response to one particular input then the system are not necessarily identical. This depends on the type of input for which the responses are same.

Consider. two systems with impulse responses.
$h_1(t) = \delta(t-1)$. and $h_2(t) = \delta(t+1)$.
for an input of $x(t) = 1$ for all t.

$y_1(t) = h_1(t) * x(t) = \delta(t-1) * x(t) = x(t-1) = \underline{1}$ .. (for all t).

$y_2(t) = h_2(t) * x(t) = \delta(t+1) * x(t) = x(t+1) = \underline{1}$ (for all t).

The output of the two systems are same for the particular input $x(t) = 1$.

For some other input, the output values are different.

for $x(t) = e^{-t}$. $\qquad y_1(t) = e^{-(t-1)} \neq e^{-(t+1)} = y_2(t)$.

∴ The systems are not identical.

However, they will be identical if the ~~input~~ response of the systems is same for impulse input.

The impulse input to system y_1 gives $\delta(t-1)$ &

the impulse response of y_2 is $\delta(t+1)$.

thus the systems in our example are different.

If the response to the one particular input is same for two systems, then they are identical if the input is impulse function.

Actually, such a property is observed for any 'invertible' input.

∴ $\quad x(t) * h_1(t) = x(t) * h_2(t)$.

$h_1(t) = h_2(t)$ if there exists $w(t)$ such that.

$w(t) * x(t) = \delta(t)$.

i.e. $\quad w(t) * x(t) * h_1(t) = w(t) * x(t) * h_2(t)$

$\delta(t) * h_1(t) = \delta(t) * h_2(t)$

$h_1(t) = h_2(t)$.

∴ The condition is there exists $w(t)$ s.t $\quad w(t) * x(t) = \delta(t)$

#3. GIVEN step response

$$g(t) = [2 - 2e^{-t/2}]\, u(t)$$

a) Find impulse response

relation between step and impulse is:

$$u(t) = \int \delta(\tau)\, d\tau$$

$$\delta(t) = u'(t)$$

$$\therefore \quad h(t) = g'(t)$$

$$= \frac{d}{dt}[2 - 2e^{-t/2}]\, u(t)$$

$$= e^{-t/2} u(t) + \underbrace{[2 - 2e^{-t/2}]\,\delta(t)}_{\text{only valid @ } t=0}$$

$$\boxed{h(t) = e^{-t/2} u(t)}$$

b) Show that the meter can be represented by $y'(t) + ay(t) = x(t)$. find a.

we know the impulse response is

$$h(t) = e^{-t/2} u(t)$$

and the relationships $y(t) \Leftrightarrow h(t)$ + $x(t) \Leftrightarrow \delta(t)$ exist. Use this in your proof.

$$LHS = y'(t) + ay(t)$$

$$= \frac{d}{dt}\left[e^{-t/2} u(t)\right] + a e^{-t/2} u(t)$$

$$= -\frac{1}{2} e^{-t/2} u(t) + e^{-t/2} \delta(t) + a e^{-t/2} u(t)$$

$$= 1 - \frac{1}{2} e^{-t/2} u(t) + a e^{-t/2} u(t)$$

$$= 1 - e^{-t/2} u(t) \left[\frac{1}{2} - a\right]$$

$$RHS = \delta(t)$$
$$= 1$$

for LHS = RHS $\boxed{a = \frac{1}{2}}$

$$\boxed{\therefore \text{ LDE is } \quad y'(t) + \frac{1}{2} y(t) = x(t)}$$

c) Devise a system that recovers input from output.

$\Rightarrow$ Find the <u>inverse</u> system

$$x(t) \to \boxed{h(t)} \to y(t) \to \boxed{h_1(t)} \to x(t)$$

we know the LDE of the original system:

$$y'(t) + \frac{1}{2} y(t) = x(t)$$

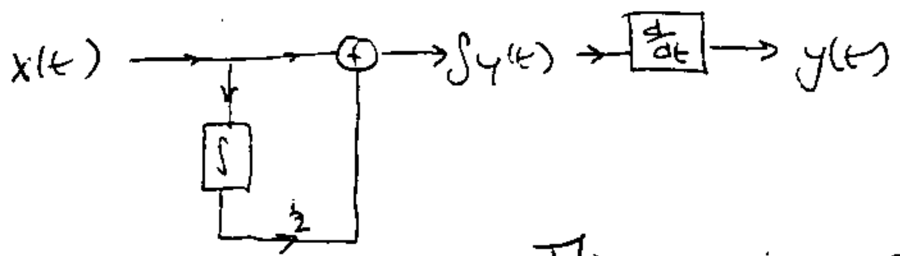
To find $h_1(t)$, the inverse system, switch x's + y's to get the inverse LDE.

$$x'(t) + \frac{1}{2} x(t) = y(t)$$

inverse system: $h_1(t) = \delta'(t) + \frac{1}{2}\delta(t)$

BLOCK DIAGRAM:

$$x'(t) + \frac{1}{2} x(t) = y(t)$$
$$x(t) + \frac{1}{2}\int x(t) = \int y(t)$$



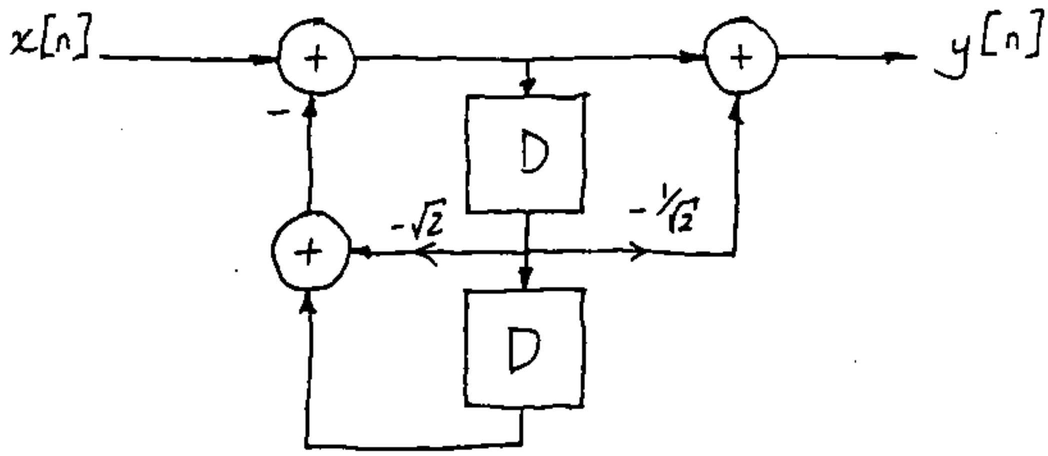
This system cannot be drawn correctly w/o a differentiator!

4. A causal discrete oscillator has the linear difference equation:

SD 252
Oct 9, 1996 Midterm

$$y[n] - \sqrt{2}\,y[n-1] + y[n-2] = x[n] - \frac{\sqrt{2}}{2}x[n-1]$$

a. Draw the Direct Form II block diagram.



b. Find the impulse response by recursively solving the LDE directly for $h(0)$, $h(1)$, $h(2)$, ..., for at least one complete period.

- Substitute $y[n] = h[n]$, $x[n] = \delta[n]$ into the L.D.E.:

$$h[n] - \sqrt{2}\,h[n-1] + h[n-2] = \delta[n] - \frac{1}{\sqrt{2}}\delta[n-1]$$

note: $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

Rearrange to isolate $h[n]$:

$$h[n] = \sqrt{2}\,h[n-1] - h[n-2] + \delta[n] - \frac{1}{\sqrt{2}}\delta[n-1].$$

Since the system is causal, we have $h[-1] = h[-2] = 0$.

$n=0$: $h[0] = \sqrt{2}(0) - 0 + 1 - \frac{1}{\sqrt{2}}(0) = 1.$

$n=1$: $h[1] = \sqrt{2}(1) - 0 - \frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}.$

$n=2$: $h[2] = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - 1 = 0$

$n=3$: $h[3] = \sqrt{2}(0) - \frac{1}{\sqrt{2}} = -\frac{1}{\sqrt{2}}$

$n=4$: $h[4] = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) - 0 = -1$

$n=5$: $h[5] = \sqrt{2}(-1) - \left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}.$

$n=6$: $h[6] = \sqrt{2}\left(-\frac{1}{\sqrt{2}}\right) - (-1) = 0$

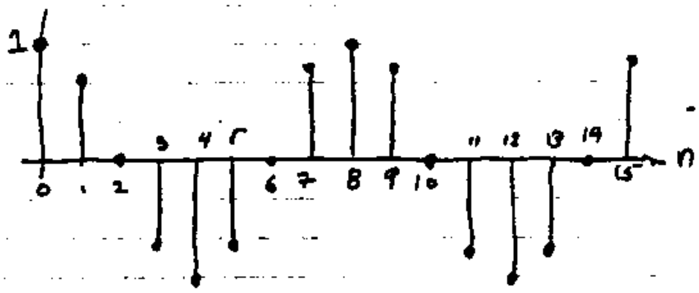
$n=7$: $h[7] = \sqrt{2}(0) - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}.$

$n=8$: $h[8] = \sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - (0) = 1.$

4. cont'd.

$$n=9: \quad h[9] = \sqrt{2}(1) - \left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{1}{\sqrt{2}}.$$

Note that $h[8]=h[0]$ and $h[9]=h[1]$; since the LDE is second order, the values of h will continue to repeat.



From this graph one might guess $h[n] = \cos\frac{\pi}{4}n\, u[n]$

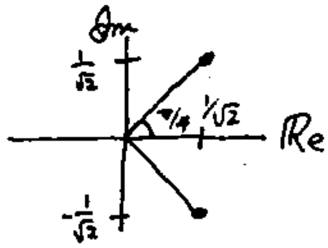
c. Find the impulse response of $y[n] - \sqrt{2}\,y[n-1] + y[n-2] = x[n]$.

- Characteristic equation:

$$1 - \sqrt{2}\,z^{-1} + z^{-2} = 0.$$

$$z^2 - \sqrt{2}\,z + 1 = 0.$$

$$z = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}j.$$



In polar form: $z = e^{\pm j\frac{\pi}{4}}$.

The homogeneous solution will be of the form

$$h_0[n] = \left[A e^{j\frac{\pi}{4}n} + B e^{-j\frac{\pi}{4}n}\right] u[n].$$

Initial conditions: use $h[n] - \sqrt{2}\,h[n-1] + h[n-2] = \delta[n]$.

From this equation, $\begin{cases} h[0] = 1. \\ h[1] = \sqrt{2}. \end{cases}$ <u>Alternate</u>: $\begin{cases} h[0] = 1 \\ h[-1] = 0. \end{cases}$

$h[0]=1$: $A + B = 1$, or $A = 1 - B$. $\quad (*)$

$h[1]=\sqrt{2}$: $A e^{j\frac{\pi}{4}} + B e^{-j\frac{\pi}{4}} = \sqrt{2}$. Substitute $(*)$:

$$(1-B)\,e^{j\frac{\pi}{4}} + B e^{-j\frac{\pi}{4}} = \sqrt{2}.$$

$$B\left(e^{-j\frac{\pi}{4}} - e^{j\frac{\pi}{4}}\right) = \sqrt{2} - e^{j\frac{\pi}{4}}.$$

$$B = \frac{\sqrt{2} - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)}{\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)} = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j}{-\sqrt{2}\,j}$$

4. cont'd.

$$B = \frac{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j}{-\sqrt{2}j} = \frac{1+j}{2}.$$

$$A = 1 - B = \frac{1-j}{2}.$$

$$\therefore\ h_0[n] = \left[\left(\frac{1-j}{2}\right)e^{j\frac{\pi}{4}n} + \left(\frac{1+j}{2}\right)e^{-j\frac{\pi}{4}n}\right]u[n].$$

d. Use linearity and time invariance to find the impulse response of the entire system.

- $h[n] = h_0[n] - \frac{1}{\sqrt{2}} h_0[n-1]$

$$= \left[\left(\frac{1-j}{2}\right)e^{j\frac{\pi}{4}n} + \left(\frac{1+j}{2}\right)e^{-j\frac{\pi}{4}n}\right]u[n] - \frac{1}{\sqrt{2}}\left[\left(\frac{1-j}{2}\right)e^{j\frac{\pi}{4}(n-1)} + \left(\frac{1+j}{2}\right)e^{-j\frac{\pi}{4}(n-1)}\right]u[n-1]$$

Consider $n \geq 1$. Then $u[n] = u[n-1] = 1$, and

$$h[n] = \frac{1-j}{2}e^{j\frac{\pi}{4}n} + \frac{1+j}{2}e^{-j\frac{\pi}{4}n} - \frac{1}{\sqrt{2}}\left(\frac{1-j}{2}\right)e^{j\frac{\pi}{4}(n-1)} - \frac{1}{\sqrt{2}}\left(\frac{1+j}{2}\right)e^{-j\frac{\pi}{4}(n-1)}$$

$$= \frac{e^{j\frac{\pi}{4}n}}{2}\left(1-j - \frac{1}{\sqrt{2}}(1-j)e^{-j\frac{\pi}{4}}\right) + \frac{e^{-j\frac{\pi}{4}n}}{2}\left(1+j - \frac{1}{\sqrt{2}}(1+j)e^{j\frac{\pi}{4}}\right)$$

$$= \frac{e^{j\frac{\pi}{4}n}}{2}\left(1-j - \frac{1}{\sqrt{2}}(1-j)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)\right) + \frac{e^{-j\frac{\pi}{4}n}}{2}\left(1+j - \frac{1}{\sqrt{2}}(1+j)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)\right)$$

Note that $1-j - \frac{1}{\sqrt{2}}(1-j)\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right) = 1-j - \frac{1}{2} + \frac{1}{2}j + \frac{1}{2}j + \frac{1}{2} = 1$

and $1+j - \frac{1}{\sqrt{2}}(1+j)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right) = 1+j - \frac{1}{2} - \frac{1}{2}j - \frac{1}{2}j + \frac{1}{2} = 1.$

So $h[n] = \frac{1}{2}\left(e^{j\frac{\pi}{4}n} + e^{-j\frac{\pi}{4}n}\right) = \cos\frac{\pi}{4}n$ for $n \geq 1$.

For $n = 0$, $h[0] = 1 = \cos\frac{\pi}{4}(0)$.

$$\therefore\ h[n] = \cos\frac{\pi}{4}n\ u[n].$$

SYDE 252 **QUIZ 1** **Oct. 9, 1997**

Aids permitted: 1 $8\frac{1}{2}$x11 page of notes, no calculators.
Each part of each problem is worth 10 marks.

1. A continuous LTI system has the LDE:

$$y''(t) + 7y'(t) + 10y(t) = 3x'(t)$$

a. Find the impulse response $h(t)$.

b. Find the output $y(t)$ if the input is the unit step, $x(t) = u(t)$. Use the graphical convolution method showing each step and carefully sketch the result.

c. Is this system stable? Explain.

2. A discrete LTI system has the LDE:

$$y(n) + \frac{16}{25}y(n-2) = x(n)$$

a. Find the impulse response $h(n)$.

b. Find the output $y(n)$ if the input is $x(n) = \delta(n) + \frac{16}{25}\delta(n-2)$.

c. Is this system invertible? Explain.

3. One straightforward strategy for finding the inverse system of simple LTI systems is to take the first few derivatives of the impulse response $h(t)$, express each in terms of $h(t)$, and try to find an LDE that $h(t)$ satisfies. Consider the noncausal system whose impulse response is:

$$h(t) = e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t).$$

a. Find and sketch $h'(t)$.

b. Find and sketch $h''(t)$.

c. What LDE does $h(t)$ satisfy?

d. What is $h_1(t)$ such that $h(t) * h_1(t) = \delta(t)$?

$\therefore$ If $x(n) = \delta(n) + \frac{16}{25}\,\delta(n-2)$

then $y(n) = h(n) + \frac{16}{25}\,h(n-2) = \underline{\delta(n)}$!

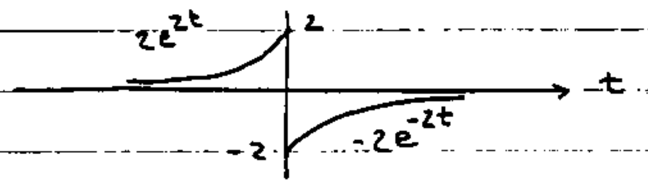
c. From b we discover that $h_1(n) = \delta(n) + \frac{16}{25}\,\delta(n-2)$

achieves $h_1(n) * h(n) = \delta(n) \Rightarrow$ invertible.

3. $h(t) = e^{-2|t|} = e^{-2t}u(t) + e^{2t}u(-t)$

a. $h'(t) = -2e^{-2t}u(t) + \cancel{e^{-2t}\delta(t)} + 2e^{2t}u(-t) - \cancel{e^{2t}\delta(t)}$

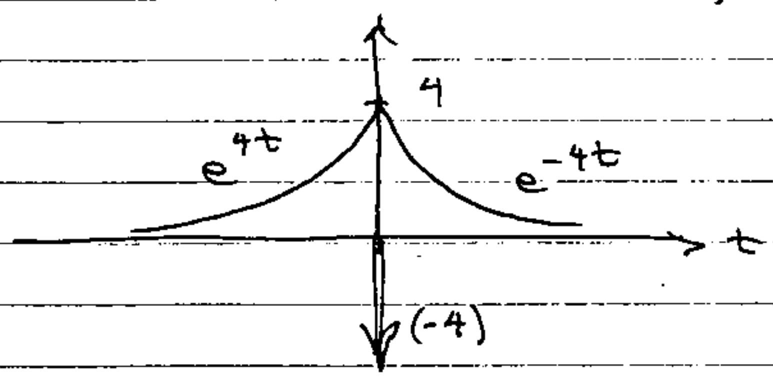
$= -2e^{-2t}u(t) + 2e^{2t}u(-t)$



b. $h''(t) = 4e^{-2t}u(t) - 2e^{-2t}\delta(t) + 4e^{2t}u(-t) - 2e^{2t}\delta(t)$

$= 4e^{-2t}u(t) + 4e^{2t}u(-t) - 4\delta(t)$

nb: $= 4h(t) - 4\delta(t)$



c. From b: $h''(t) - 4h(t) = -4\delta(t)$

$y''(t) - 4y(t) = -4x(t)$

d. From c: $-\frac{1}{4} h''(t) + h(t) = \delta(t)$

$$\therefore \quad h(t) * \left(\delta(t) - \frac{1}{4}\delta''(t)\right) = \delta(t)$$

$$\therefore \quad h_1(t) = \delta(t) - \frac{1}{4}\delta''(t) = \delta(t) - \frac{1}{4} u_2(t)$$

SYDE 252 QUIZ 1 Oct. 15, 1998

Prof. Ed Jernigan

Aids permitted: 1 $8\frac{1}{2}$x11 page of notes, no calculators.

Each part of each problem is worth 10 marks.

1. A continuous LTI system has the LDE:

$$y''(t) + 2y'(t) + 2y(t) = x(t)$$

   a. Find the response of the system if the input is $\delta(t)$.

   b. Find the output $y(t)$ if the input is the unit step, $x(t) = u(t)$.

   c. Find the output $y(t)$ if the input is $\delta'(t)$, the unit doublet.

   d. Find the impulse response if the right hand side of the equation is $x(t) + x'(t)$.

2. A discrete LTI oscillator of period 4 has the LDE:

$$y(n) + y(n-2) = x(n)$$

   a. Find the impulse response $h(n)$.

   b. Find the output $y(n)$ if the input is $x(n) = u(n)$.

   c. Is there any bounded input which produces an unbounded output? Explain. Give an example if possible.

3. Suppose you graduate with a total debt of \$10,000. If you negotiate a deal with your banker to repay the loan over 5 years in 60 equal monthly payments, with an interest of 0.5% monthly on the unpaid balance, what will your monthly payments be? The parts of the problem will help you formulate this problem as a discrete system.

   a. Explain how your indebtedness satisfies the equation $y(n) - ay(n-1) = x(n)$. What are $x(n), y(n),$ and a?

   b. If you borrow \$10,000 at month 0 and make payments of \$P monthly beginning at month 1, you can express the input to your debt system as $x(n) = 10{,}000\delta(n) - Pu(n-1)$. Find $y(n)$, your debt at month n.

   c. What are your monthly payments if you are to be free of debt after 5 years? (Express the payment amount P in terms of a.)

SD252 Quiz 1 Solutions

F '98

26¢ HTAX

SD252 QUIZ1

Oct 15

1. $y''(t) + 2y'(t) + 2y(t) = x(t)$

a. $x(t) = \delta(t) \Rightarrow y(t) = h(t)$, homogeneous sol'n w/ $h'(0)=1$, $h(0)$

$s^2 + 2s + 2 = 0 \qquad s = -1 \pm j \qquad h(t) = A e^{(-1+j)t} + B e^{(-1-j)}$

$$\left.\begin{aligned} h'(0) &= (-1+j)A - (1+j)B = 1 \\ h(0) &= A + B = 0 \end{aligned}\right\} \begin{aligned} &A(-1+j+1+j) = 1 \\ &B = -A \end{aligned} \qquad \begin{aligned} A &= 1/2j \\ B &= -1/2 \end{aligned}$$

$$h(t) = \frac{1}{2j} e^{-t}(e^{jt} - e^{-jt})u(t) = e^{-t}\sin t\, u(t)$$

b. $x(t) = u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \quad \therefore \quad y(t) = \int_{-\infty}^{t} h(\tau)d\tau \quad (= u(t) * h(t))$

$$y(t) = \int_0^t \frac{1}{2j}\left(e^{(-1+j)\tau} - e^{-(1+j)\tau}\right)d\tau$$

$$= \frac{1}{2j}\left[\frac{1}{-1+j}\left(e^{(-1+j)t} - 1\right) + \frac{1}{1+j}\left(e^{-(1+j)t} - 1\right)\right]$$

$$= \frac{1}{2j}\left[\frac{-1-j}{2}\left(e^{(-1+j)t} - 1\right) + \frac{1-j}{2}\left(e^{-(1+j)t} - 1\right)\right]$$

$$= \frac{1}{2j}\left[-\frac{1}{2}\left(e^{-(1+j)t} - e^{-(1+j)t}\right) - \frac{j}{2}\left(e^{(-1+j)t} + e^{-(1+j)t}\right) + j\right]$$

$$= \left[\frac{1}{2} - \frac{1}{2}e^{-t}(\sin t + \cos t)\right]u(t)$$

c. $x(t) = \delta'(t) \quad \therefore \quad y(t) = h'(t)$

$$y(t) = -e^{-t}\sin t\, u(t) + e^{-t}\left(\cos t\, u(t) + \overbrace{\sin t\, \delta(t)}^{0\,\delta(t)}\right)$$

$$y(t) = e^{-t}(\cos t - \sin t)\,u(t)$$

d. If RHS is $x(t) + x'(t)$, $h(t) = h_a(t) + h_a'(t)$

$h(t) = h_a(t) + h_c(t)$ (from parts a & c !)

$= e^{-t}\sin t\, u(t) + e^{-t}\cos t\, u(t) - e^{-t}\sin t\, u(t)$

$h(t) = e^{-t}\cos t\, u(t)$

2. $y(n) + y(n-2) = x(n)$

a. $h(n)$: $1 + z^{-2} = 0 \Rightarrow z^2 + 1 = 0 \Rightarrow z = \pm j = e^{\pm j \frac{\pi}{2}}$

$h(-1) = 0 = Aj^{-1} + B(-j)^{-1} = A/j - B/j = 0$ $\quad A = B$

$h(0) = 1 = A + B$ $\quad A = B = 1/2$

$h(n) = \left[\frac{1}{2} e^{j\frac{\pi}{2}n} + \frac{1}{2} e^{-j\frac{\pi}{2}n}\right] u(n) = \cos\frac{\pi}{2}n \; u(n)$

b. $y(n) = u(n) * h(n) = \sum_{k=0}^{n} \left(\frac{1}{2} e^{j\frac{\pi}{2}k} + \frac{1}{2} e^{-j\frac{\pi}{2}k}\right)$

$y(n) = \left[\frac{1}{2}\left(\frac{1 - e^{j\frac{\pi}{2}(n+1)}}{1 - e^{j\pi/2}}\right) + \frac{1}{2}\left(\frac{1 - e^{-j\frac{\pi}{2}(n+1)}}{1 - e^{-j\pi/2}}\right)\right] u(n)$

$= \frac{1}{2}\left[\frac{1}{1-j}\left(1 - e^{j\frac{\pi}{2}(n+1)}\right) + \frac{1}{1+j}\left(1 - e^{-j\frac{\pi}{2}(n+1)}\right)\right]$

$= \frac{1}{2}\left[\frac{1+j}{2}\left(1 - e^{j\frac{\pi}{2}(n+1)}\right) + \frac{1-j}{2}\left(1 - e^{-j\frac{\pi}{2}(n+1)}\right)\right] u(n)$

$= \frac{1}{2}\left[1 + \sin\left(\frac{\pi}{2}(n+1)\right) - \cos\left(\frac{\pi}{2}(n+1)\right)\right] u(n)$

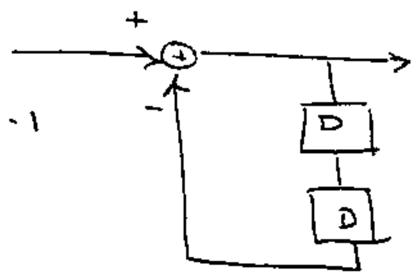
(other forms possible)

c. Since $h(n) = \cos\frac{\pi}{2}n\ u(n) = \{1\ 0\ -1\ 0\ 1\ 0\ -1\ 0 \ldots$

$\sum_n |h(n)| \to \infty$, $h(n)$ is not absolutely summabl[e]
the system is therefore unstable and there must be some bounded input that produces an unbounded output. One such bounded input would be $\cos\frac{\pi}{2}n\ u(n)$ itself. Either convolve $\cos\frac{\pi}{2}n\ u(n) * \cos\frac{\pi}{2}n\ u(n)$ or consider the block diagram to see that the response to $\cos\frac{\pi}{2}n\ u(n)$ grows without bound:

$\cos\frac{\pi}{2}n$

input 1 0 -1 0 1 0 -1



output 1 0 -2 0 3 0 -4
0 1 2 3 4 5 6

and you can show $y(n) = \frac{n+2}{2} \cos\frac{\pi}{2}n\ u(n)$

3. a. \$ owed @ month n = \$ owed @ month n-1
+ interest − payment

$\Rightarrow\ y(n) = y(n-1) + r\,y(n-1) + x(n)$

w/ r = % interest = .005 $\quad x(n) = -$payment

$\therefore\ y(n) - a\,y(n-1) = x(n)$

w/ $a = 1 + r = 1.005 \quad x(n) = -P\,u(n-1)$
for P \$ paid per month beginning at $n=1$

b. $y(n) - a\,y(n-1) = x(n)$ w/ $x(n) = 10000\,\delta(n) - P\,u(n-1)$

$\Rightarrow y(n) = x(n) * h(n)$ w/ $h(n) = a^n u(n)$ from LDE

Since $x(n) = 10000\,\delta(n) - P\,u(n-1)$, find $h(n) * u(n)$ and use LSI properties

$$u(n) * h(n) = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1-a}\, u(n)$$

Then $y(n) = 10{,}000\, h(n) - P\, u(n-1) * h(n)$

$$= 10{,}000\, a^n - P\left(\frac{1-a^n}{1-a}\right) \quad \text{for } n \geq 1$$

$$y(n) = 10{,}000\,\delta(n) + \left[10000\, a^n - P\left(\frac{1-a^n}{1-a}\right)\right] u(n-1)$$

c. Find P by requiring $y(60)$ (or $y(61)$ depending on your interpretation of "after 5 years") to be zero

$$\Rightarrow 10{,}000\, a^n - P\left(\frac{1-a^n}{1-a}\right) = 0 \quad @\ n = 60$$

$$P = 10{,}000\, a^n \frac{1-a}{1-a^n} \quad \text{w/ } a = 1.005$$

$$P = 10000 \frac{a-1}{1-a^{-60}} = \frac{50}{1-a^{-60}}$$

$P \cong \$193.33$ if we had a calculator

SYDE 252 QUIZ 1 Oct. 14, 1999

Prof. Ed Jernigan

Aids permitted: 1 $8\frac{1}{2}$x11 page of notes, no calculators.

Each part of each problem is worth 10 marks.

1. A continuous LTI system has the impulse response:

$$h(t) = e^{-a|t|} = \begin{cases} e^{at} & t < 0 \\ e^{-at} & t > 0 \end{cases}; \quad \text{assume } a > 0.$$

   a. Find the response of the system if the input is the unit step, $u(t)$.
   b. Find the output $y(t)$ if the input is $u_2(t) = \delta''(t)$.
   c. Find the output $y(t)$ if the input is, $x(t) = e^{-b|t|}$, $0 < b < a$.

2. Suppose a discrete LTI system has the impulse response:

$$h(n) = u(n) - 2u(n-3) + u(n-6).$$

   a. Find the output $y(n)$ if the input is $x(n) = u(n)$.
   b. Find the output $y(n)$ if the input is $x(n) = a^n u(n)$, $0 < a < 1$.
   c. Find the output $y(n)$ if the input is $x(n) = u(-n) - 2u(3-n) + u(6-n)$.

3. A continuous LTI system has the LDE:

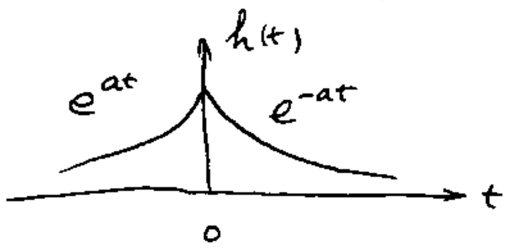
$$y''(t) + 2y'(t) + 2y(t) = x(t)$$

   a. Find the impulse response of the system, $h(t)$.
   b. Find the impulse response if the right hand side of the equation is $x''(t)$.

4. Suppose an LTI system has the LDE: $y(n) + by(n-1) + cy(n-2) = x(n)$.

   a. What conditions on b and c produce an impulse response which is a decaying sinusoid? A growing sinusoid? A constant amplitude sinusoid? Explain.
   b. Find the impulse response if $b = -1$, $c = \frac{1}{2}$.

1 $h(t) = e^{-a|t|}$



a) $y(t) = u(t) * h(t)$

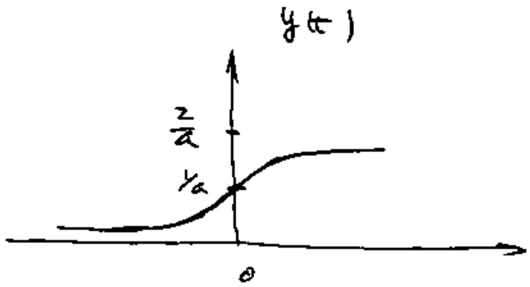
$$= \int_{-\infty}^{t} e^{-a|\tau|} d\tau$$

I: $t < 0$ $\quad y(t) = \int_{-\infty}^{t} e^{a\tau} d\tau = \frac{1}{a} e^{at}$

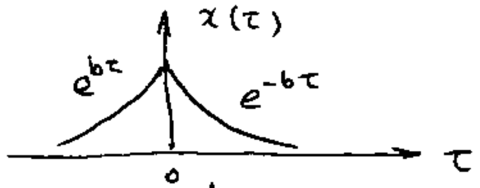
II: $t > 0$ $\quad y(t) = \int_{-\infty}^{0} e^{a\tau} d\tau + \int_{0}^{t} e^{-a\tau} d\tau$

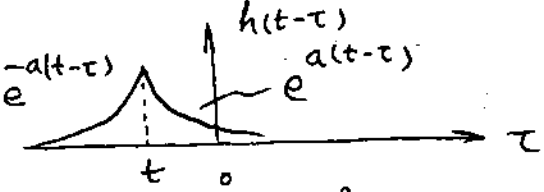
$$= \frac{1}{a} + \frac{1}{a} - \frac{1}{a} e^{-at} = \frac{2}{a} - \frac{1}{a} e^{-at}$$

$$\therefore\ y(t) = \begin{cases} \frac{1}{a} e^{at}, & t < 0 \\ \frac{1}{a}(2 - e^{-at}), & t > 0 \end{cases}$$



b) $x(t) = e^{-b|t|}$, $b < a$





for $t < 0$, $y(t) = \int_{-\infty}^{t} e^{b\tau} e^{-a(t-\tau)} d\tau + \int_{t}^{0} e^{b\tau} e^{a(t-\tau)} d\tau + \int_{0}^{\infty} e^{-b\tau} e^{a(t-\tau)} d\tau$

$$= e^{-at} \int_{-\infty}^{t} e^{(a+b)\tau} d\tau + e^{at} \int_{t}^{0} e^{(b-a)\tau} d\tau + e^{at} \int_{0}^{\infty} e^{-(a+b)\tau} d\tau$$

$$= e^{-at} \frac{e^{(a+b)t}}{a+b} + e^{at} \frac{1 - e^{(b-a)t}}{b-a} + e^{at} \frac{1}{a+b}$$

$$y(t) = \left(\frac{1}{a+b} + \frac{1}{b-a}\right)e^{at} + \left(\frac{1}{a+b} - \frac{1}{b-a}\right)e^{bt}$$

$$= \frac{2a}{a^2-b^2}e^{bt} - \frac{2b}{a^2-b^2}e^{at}$$

Because of symetry, when $y>0$, we should have

$$y(t) = \frac{2a}{a^2-b^2}e^{-bt} - \frac{2b}{a^2-b^2}e^{-at}$$

So, finally

$$y(t) = \frac{2a}{a^2-b^2}e^{-b|t|} - \frac{2b}{a^2-b^2}e^{-a|t|}$$

c) $x(t) = u_2(t) = \delta''(t) \implies y(t) = h''(t)$

$$h(t) = e^{at}u(-t) + e^{-at}u(t)$$

$$h'(t) = ae^{at}u(-t) - \cancel{e^{at}\delta(t)} - ae^{-at}u(t) + \cancel{e^{-at}\delta(t)}$$

$$= ae^{at}u(-t) - ae^{-at}u(t)$$

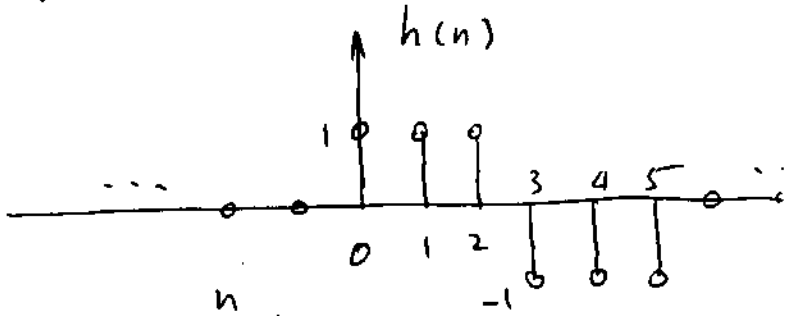
$$h''(t) = a^2e^{at}u(-t) - ae^{at}\delta(t) + a^2e^{-at}u(t) - ae^{-at}\delta(t)$$

$$= a^2e^{at}u(-t) + a^2e^{-at}u(t) - 2a\delta(t)$$

$$= a^2h(t) - 2a\delta(t)$$

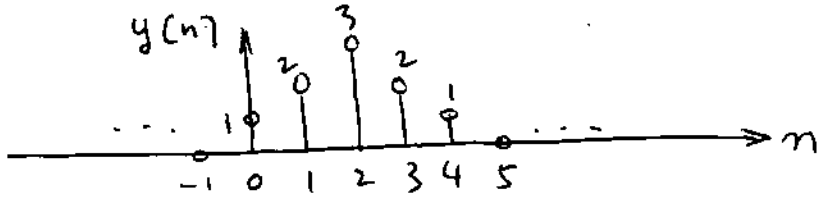
or $y(t) = a^2e^{-a|t|} - 2a\delta(t)$

2 $h[n] = u(n) - 2u(n-3) + u(n-6)$



a) $x[n] = u(n) \Rightarrow y[n] = \sum_{k=-\infty}^{n} h[k]$

$\therefore y[n] = \{ \cdots 0 \ \underset{\uparrow}{1} \ 2 \ 3 \ 2 \ 1 \ 0 \cdots \}$
0



b) $x[n] = a^n u(n)$

$n < 0 \qquad y[n] = 0$

$0 \le n \le 2 \qquad y(n) = \sum_{k=0}^{n} a^k = \dfrac{1-a^{n+1}}{1-a} = \begin{cases} 1, & n=0 \\ 1+a, & n=1 \\ 1+a+a^2, & n=2 \end{cases}$

$3 \le n \le 5 \qquad y[n] = -\sum_{k=0}^{n-3} a^k + \sum_{k=n-2}^{n} a^k$

$= a^{n-2}\left(\dfrac{1-a^3}{1-a}\right) - \dfrac{1-a^{n-2}}{1-a}$

$= \dfrac{2a^{n-2} - a^{n+1} - 1}{1-a}$

$n \geq 5$ $\quad y[n] = \sum_{k=n-2}^{n} a^k - \sum_{k=n-5}^{n-3} a^k$

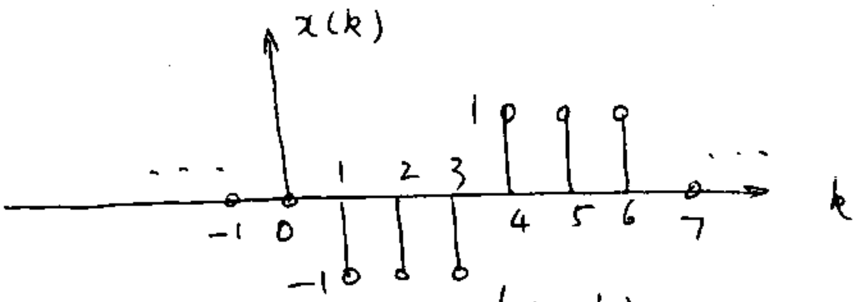
$$= a^{n-2} \sum_{k=0}^{2} a^k - a^{n-5} \sum_{k=0}^{2} a^k$$

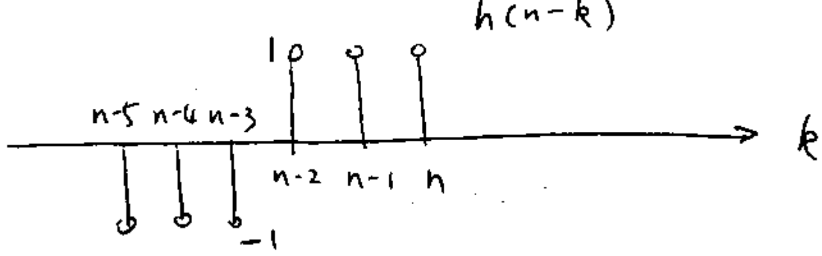
$$= a^{n-2} \frac{1-a^3}{1-a} - a^{n-5} \frac{1-a^3}{1-a}$$

$$= \frac{1-a^3}{1-a} \left(a^{n-2} - a^{n-5} \right)$$

$$= -a^{n-5} \frac{(1-a^3)^2}{1-a}$$

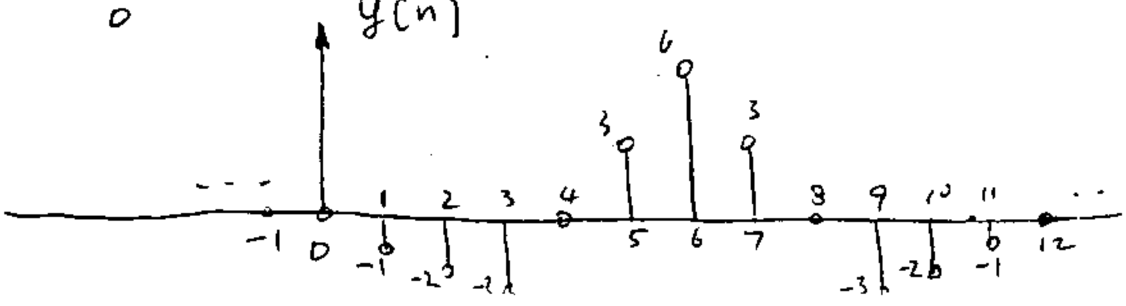
c) $\quad x(n) = u(-n) - 2u(3-n) + u(6-n)$





$y[n] = x(n) * h(n)$

$$= \{ \cdots \underset{\substack{\uparrow \\ 0}}{0} \; -1 \; -2 \; -3 \; 0 \; 3 \; 6 \; 3 \; 0 \; -3 \; -2 \; -1 \; 0 \cdots \}$$



3. $y''(t)+2y'(t)+y(t)=x(t)$

a) $s^2+2s+2=0 \Longrightarrow s=-1\pm j$

$$h(t)=Ae^{-t}e^{jt}+Be^{-t}e^{-jt} \Rightarrow A+B=0 \ @\ t=0$$

$$h'(t)\Big|_{t=0}=(-1+j)A-(1+j)B=1$$

$$\therefore\ A=\frac{1}{2j}\ ,\quad B=-\frac{1}{2j}$$

$$\therefore\ h(t)=e^{-t}\sin t\ u(t)$$

b) $x''(t)$ at RHS $\Longrightarrow h(t)=h_o''(t)$

where we have got $h_o(t)=e^{-t}\sin t\, u(t)$

$$h_o'(t)=-e^{-t}\sin t\ u(t)+e^{-t}\cos t\, u(t)+e^{-t}\cancelto{0}{\sin t\,\delta(t)}$$

$$=e^{-t}(\cos t-\sin t)\,u(t)$$

$$h(t)=h_o''(t)=-e^{-t}(\cos t-\sin t)\,u(t)$$
$$+e^{-t}(-\sin t-\cos t)\,u(t)$$
$$+e^{-t}(\cos t-\sin t)\,\delta(t)$$

$$=\delta(t)-2e^{-t}\cos t\ u(t)$$

4. $y[n] + b\,y(n-1) + c\,y(n-2) = x[n]$

a) $z^2 + bz + c = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4c}}{2} = z_1,\ z_2$$

– Sinusoid if z_1, z_2 are complex $\Rightarrow 4c > b^2$

– decays if $|z| = \sqrt{\frac{b^2 + (4c - b^2)}{2}} = \sqrt{c} < 1$

or $c < 1$

– grows if $c > 1$

– constant if $c = 1$

b) for $b = -1$, $c = \frac{1}{2}$ $\Rightarrow z = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1}{2} \pm \frac{j}{2}$

$$= \frac{\sqrt{2}}{2} e^{\pm j\frac{\pi}{4}}$$

$$h(n) = A\left(\frac{\sqrt{2}}{2}\right)^n e^{j\frac{\pi}{4}n} + B\left(\frac{\sqrt{2}}{2}\right)^n e^{-j\frac{\pi}{4}n}$$

By $\begin{cases} h(-1) = A\left(\frac{\sqrt{2}}{2}\right)^{-1} e^{-j\frac{\pi}{4}} + B\left(\frac{\sqrt{2}}{2}\right)^{-1} e^{j\frac{\pi}{4}} = 0 \\ h(0) = A + B = 1 \end{cases}$

we can get $A = \frac{1}{1+j} = \frac{1-j}{2}$, $B = \frac{1+j}{2}$

$\therefore h(n) = \left(\frac{\sqrt{2}}{2}\right)^n \left(\cos\frac{\pi}{4}n + \sin\frac{\pi}{4}n\right) u(n)$

or, $h(n) = \left(\frac{1}{2}\right)^{n/2} \left(\cos\frac{\pi}{4}n + \sin\frac{\pi}{4}n\right) u(n)$ (decays!)