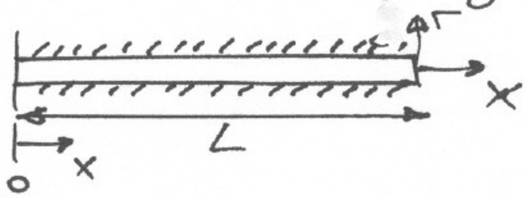
Seperation of Veriebles

Motivation Example : Transient heat transfer along a thin rod (cylinder)



Rod is very thin and insulated around
$\Rightarrow$ no gradients in the r-direction

Solve $T(x,t)$ BC's: $\begin{cases} T(0,t) = 0 \\ T(L,t) = 0 \end{cases}$ I.C. $T(x,t) = f(x)$

Energy Balance, Transient, 1-D

$$\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \qquad \frac{k}{\rho c_p} \triangleq \alpha \tag{1}$$

Let $\alpha = c^2$ for convenience

From (1) $$\frac{\partial T}{\partial t} = c^2 \frac{\partial^2 T}{\partial x^2} \tag{2}$$

Assume solution $$T(x,t) = F(x)\,G(t) \tag{3}$$

Substitute (3) into (2):

$$F \frac{\partial G}{\partial t} = c^2 G \frac{\partial^2 F}{\partial x^2} \quad \text{or} \quad F G' = c^2 G F'' \tag{4}$$

From (4)

$$\frac{G'}{c^2 G} = \frac{F''}{F} = \text{const} \qquad (5)$$

Assume a positive separation constant p^2

From (5)

$$G' - p^2c^2G = 0 \quad \text{and} \quad F'' - p^2F = 0 \qquad (6)$$

$$G' - p^2c^2G = 0 \quad \Rightarrow \quad G = c_1 e^{p^2c^2t} \qquad (7)$$

$$F'' - p^2F = 0 \quad \Rightarrow \quad F = c_2e^{px} + c_3e^{-px} \qquad (8)$$

From (7) and (8) and (3)

$$T(x,t) = c_1 e^{p^2c^2t}\left(c_2e^{px} + c_3e^{-px}\right) \qquad (9)$$

However $T(x,t) \to \infty$ for $t \to \infty \Rightarrow$ impossible

Can the separation constant be zero? $(p^2 = 0)$

From (5) $\quad G' = 0 \quad F'' = 0$

$$\Rightarrow \quad G = \text{const} \qquad F = \int_0^x F''\,dx$$

$\Rightarrow$ From (3) $\quad T(x,t) = \text{const}\left(\int_0^x F''\,dx\right) \neq f(t)$ impossible

Then, const in (5) must be negative $(-p^2)$

From (5) $\quad G' + p^2c^2G = 0 \qquad G = De^{-p^2c^2t}$

$$F'' + p^2F = 0 \qquad F = A\cos px + B\sin px$$

from (3) $\quad T(x,t) = De^{-p^2c^2t}\left(A\cos px + B\sin px\right) \qquad (10)$

So far is o.k. T is a function of x, t and

goes to 0 for $t \to \infty$ (steady state)

Now, substitute boundary conditions:

@ $x=0$, $T=0 = De^{-p^2c^2t}(A \cdot 1 + B \cdot 0) \Rightarrow A=0$

@ $x=L$, $T=0 = De^{-p^2c^2t}\, B\sin pL \Rightarrow p_nL = n\pi$

$n = 0, 1, \ldots$

$\Rightarrow$ from (3) $\quad T(x,t) = \sum_{n=0}^{\infty} E_n e^{-p_n^2c^2t} \sin \frac{n\pi}{L}x \qquad (11)$

And finally, initial condition:

$$T(x,0) = f(x) = \sum_{n=0}^{\infty} E_n \sin \frac{n\pi}{L}x \qquad (12)$$

Also $\lambda_n = \frac{n\pi}{L}$ eigenvalues and $\sin(\lambda_n x)$ eigenfunctions
To equate LHS with RHS is required to represent $f(x)$ as a series of sines

$\Downarrow$

Fourier Series

Fourier Series

Assume, we want to represent a periodic function $f(x)$ (w/ period 2π) by:

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

or in compact form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Question: what are the coefficients $a_0, a_1, b_1, \ldots$ etc to satisfy this equation ??

Computation of a_0

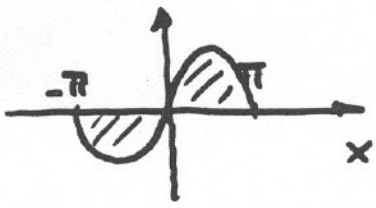
if $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

Integrate both sides by dx ($-\pi$ to π)

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \int_{-\pi}^{\pi} \sum_{1}^{\infty} (a_n \cos nx + b_n \sin nx)\,dx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dX + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx\,dX + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx\,dX$$

$n=1$



$\sin X$

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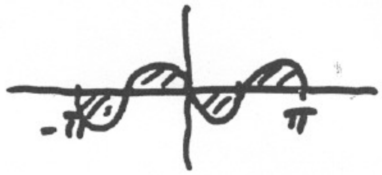
$$\int_{-\pi}^{\pi} \sin x\,dX = 0$$

$n=2$



$\sin 2X$

$$\int_{-\pi}^{\pi} \sin 2x\,dX = 0$$

In the same way is possible to show

$$\int_{\pi}^{\pi} \cos x = \int_{-\pi}^{\pi} \cos 2x = \dots = 0$$

Then

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx = a_0(2\pi)$$

$$\Rightarrow a_0 = \frac{\int_{-\pi}^{\pi} f(x)\,dx}{2\pi}$$

<u>Calculation of a_n</u> (for $n > 0$)

From the original equation!

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Multiply both sides by $\cos mx$ and integrate between $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} a_0 \cos mx\,dx + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx$$

$$+ \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx =$$

Apply trigonometric formulae; e.g.

$$\cos nx \cos mx = \frac{1}{2}\left[\cos(n+m)x + \cos(n-m)x\right]$$

$$\sin nx \cos mx = \frac{1}{2}\left[\sin(n+m)x + \sin(n-m)x\right]$$

and using identities:

$$\int_{-\pi}^{\pi} \cos(n+m)x\,dx = 0 \qquad \int_{-\pi}^{\pi} \sin(n+m)x\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin(n-m)x\,dx = 0$$ and,

$$\int_{-\pi}^{\pi} \cos(n-m)x\,dx = \begin{cases} 2\pi & n = m \\ 0 & n \neq m \end{cases}$$

Then, we obtain

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, \ldots$$

In the same way, we can show, by multiplying by $\sin mx$ and $\int_{-\pi}^{\pi}$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, \ldots$$

<u>Summary</u>

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

called
Euler formulae

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If the function has period $2L$ instead of 2π, what are the formulae for a_0, a_n and b_n?

Use the substitution

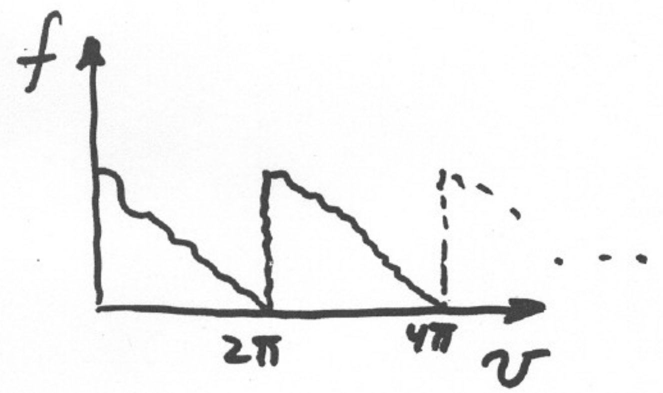
$$v = \frac{\pi x}{L} \Rightarrow x = \frac{Lv}{\pi} \Rightarrow dx = \frac{L}{\pi}\,dv$$

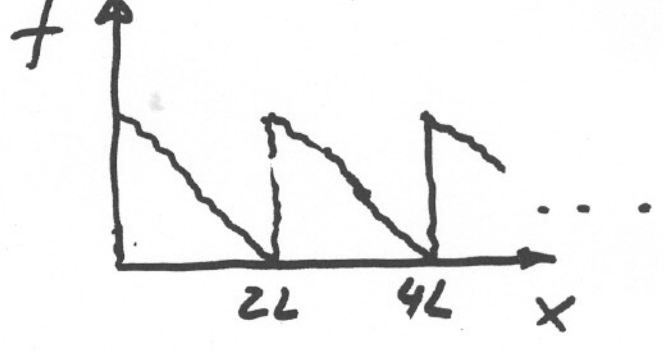
So, for example if

$f(v)$ is periodic with period 2π with respect to v

$f\left(\frac{\pi x}{L}\right)$ is periodic with period $2L$ with respect to x

$\downarrow$





After substitution of $v = \frac{\pi x}{L}$

$$dv = \frac{\pi}{L} dx$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} \underbrace{f(x)}_{\text{periodicity } 2L} dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$