Example: Fourier Series

Apply the Fourier Series Method to represent the following function:

\[ f(x) \]

-2l  -l  0  l  2l

\[ -x \]

"Periodic Square Wave"

\[ a_0 = \frac{1}{2l} \int_{-l}^{l} f(x) \, dx = \frac{1}{2l} \int_{-l}^{0} (-k \, dx) + \frac{1}{2l} \int_{0}^{l} k \, dx = \]

\[ = -\frac{k}{2l} l + \frac{k}{2l} l = 0 \]

\[ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx = \frac{1}{l} \int_{-l}^{0} (-k \cos \frac{n\pi x}{l} \, dx \]

\[ + \frac{1}{l} \int_{0}^{l} k \cos \frac{n\pi x}{l} \, dx \]
\[ a_n = -\frac{K}{\ell} \left[ \frac{\ell}{n\pi} \sin \frac{n\pi x}{\ell} \right]^0_0 + \frac{K}{\ell} \left[ \frac{\ell}{n\pi} \sin \frac{n\pi x}{\ell} \right]^l_0 \]

\[ = 0 \]

\[ b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell} \, dx = \]

\[ = \frac{1}{\ell} \int_{-\ell}^{\ell} (-K) \sin \frac{n\pi x}{\ell} \, dx + \frac{1}{\ell} \int_{-\ell}^{\ell} K \sin \frac{n\pi x}{\ell} \, dx \]

\[ = \frac{K}{\ell} \left[ \frac{\ell}{n\pi} \cos \frac{n\pi x}{\ell} \right]^0_0 - \frac{K}{\ell} \left[ \frac{\ell}{n\pi} \cos \frac{n\pi x}{\ell} \right]^l_0 \]

\[ = \frac{K}{n\pi} \left( 1 - \cos \frac{n\pi}{l} \right) - \frac{K}{n\pi} \left( \cos \frac{n\pi}{l} - 1 \right) \]

\[ = \frac{2K}{n\pi} - \frac{2K}{n\pi} \cos \frac{n\pi}{l} \]

\( b_n \quad \text{for } n = 2, 4, \ldots \)

\[ b_n \quad \frac{4K}{n\pi} \quad \text{for } n = 1, 3, 5, \ldots \]
$f(x) = \frac{4K}{\pi} \sin \frac{\pi x}{l} + \frac{4K}{3\pi} \sin \frac{3\pi x}{l} + \frac{4K}{5\pi} \sin \frac{5\pi x}{l} + \cdots$

1st term

Sum of first 2 terms

Sum of first 3 terms
Simplification of the method

In the square wave example, $a_0$ and $a_n$ are zero.

Could we predict these results without performing the corresponding integrations?

Yes

It depends on the function $f(x)$ being **ODD** or **EVEN**

\[
\text{EVEN } \left( \int_{-\ell}^{\ell} f(x) dx = 2 \int_{0}^{\ell} f(x) dx \right)
\]

\[
\text{ODD } \left( \int_{-\ell}^{\ell} f(x) dx = 0 \right)
\]
The product

odd function \times even function = odd function

Now, look at the Fourier coefficients:

Assume \( f \) is even

\[
bn = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx = 0
\]

\[
\begin{array}{c}
\text{even} \\
\text{odd}
\end{array}
\]

\[
\begin{array}{c}
\text{odd} \\
\text{even}
\end{array}
\]

\[
an \neq 0
\]

\( \Rightarrow \) \( f \) will be represented by series of cosines

Assume \( f \) is odd (e.g. square wave)

\[
am = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx = 0
\]

\[
\begin{array}{c}
\text{odd} \\
\text{even}
\end{array}
\]

\[
bn \neq 0
\]

\( \Rightarrow \) \( f \) represented by series of sines.
Orthogonality of Functions

In the previous section (Fourier Series) we have used several times the following property:

\[
\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \begin{cases} 0 & n = m, \text{ constant } \neq 0 \\ \pi & n \neq m 
\end{cases}
\]

\[
\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \, dx = \begin{cases} 0 & n = m, \text{ constant } \neq 0 \\ \pi & n \neq m 
\end{cases}
\]

Then \( \sin \left( \frac{n\pi x}{L} \right) \) and \( \sin \left( \frac{m\pi x}{L} \right) \) are orthogonal.

In the same way \( \cos \left( \frac{n\pi x}{L} \right) \) and \( \cos \left( \frac{m\pi x}{L} \right) \) are orthogonal.

Proof:

\[
\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \, dx = \frac{1}{2} \left[ \cos \left( \frac{(n-m)\pi x}{L} \right) - \cos \left( \frac{(n+m)\pi x}{L} \right) \right] \, dx
\]

\[
\begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m 
\end{cases}
\]

In general two functions \( f_n(x) \) and \( f_m(x) \) are said to be orthogonal iff:

\[
\int_{a}^{b} g(x) f_n(x) f_m(x) \, dx = 0 \quad \text{if} \quad m \neq n
\]

\[
\neq 0 \quad \text{if} \quad m = n
\]

where \( g(x) \) is a weighting function.
Completion of the 1-D Rod Heat Transfer Problem

From (11) before, the general solution is:

\[ T(x,t) = \sum_{n=0}^{\infty} E_n e^{-\frac{\alpha^2 c^2 t}{L}} \sin \frac{n\pi x}{L} \]  

(11)

after substitution of initial condition:

\[ T(x,0) = f(x) = \sum_{n=0}^{\infty} E_n \sin \frac{n\pi x}{L} \]  

(12)

The idea is to represent \( f(x) \) as a series of sines and cosines (Fourier series) and then equate coefficients in (12).

So, expanding \( f(x) \) as a Fourier series, in general:

\[ f(x) = a_0 + \sum_{n=0}^{\infty} (a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}) \]  

(13)

To be able to satisfy (12) \( a_0, a_n = 0 \), \( \Rightarrow \)

If \( f \) is odd function \( \sin \) \(-L \leq x < L\)  

\( \Rightarrow \) to obtain a series of sines only.
\[ b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \quad (14) \]

From (12)
\[ f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n \pi x}{L} = \sum_{n=0}^{\infty} E_n \sin \frac{n \pi x}{L} \quad (15) \]

Thus, from (14) and (15) \[ E_n = b_n \]

Thus, the solution is from (11) and (14) and (15)
\[ T(x, t) = \sum_{n=0}^{\infty} \left( \frac{2}{L} \int_0^L f(x) \sin \frac{n \pi x}{L} \, dx \right) \sin \frac{n \pi x}{L} \quad (16) \]

Simple Example:

Initial Condition \[ f(x) = \alpha \sin \frac{n \pi x}{L} = T(x, t=0) \]

From (14)
\[ E_n = b_n = \frac{2}{L} \int_0^L \alpha \sin \frac{n \pi x}{L} \sin \frac{n \pi x}{L} \, dx \quad (17) \]

Easy to show that

\[ E_n = 0 \quad n \neq 1 \quad \text{because} \quad \int \sin \frac{n \pi x}{L} \sin \frac{n \pi x}{L} \, dx = 0 \quad \text{for } n \neq 1 \]

\[ n = 1 \quad E_1 = \frac{2 \alpha}{L} \int_0^L \sin^2 \frac{n \pi x}{L} \, dx = \frac{2 \alpha}{L} \int_0^L \left( \frac{1}{2} - \frac{1}{2} \cos \frac{2n \pi x}{L} \right) \, dx = \alpha \]
Then
\[ T(x, t) = \alpha e^{-\frac{\pi^2}{L^2} \frac{e^{2t}}{L^2}} \sin \frac{\pi x}{L} \]

\[ \lim_{t \to \infty} T(x, t) = 0 \]

The solution can be easily obtained by inspection.

From (12) above

\[ \alpha \sin \frac{\pi x}{L} = \sum_{n=0}^{\infty} E_n \sin \frac{n \pi x}{L} \]

\[ \Rightarrow E_1 = \alpha \]

\[ E_2 = E_3 = \ldots = 0 \]