Solution of PDE's using Laplace

When?
- Problem cannot be solved with separation of variables
- Transform with respect to variable that goes from 0 → ∞
e.g. time (0 < t < ∞)

Motivation Example

Solve the temperature $T(x,t)$ in a semi-infinite well (very thick wall)

Heat transfer in 1 dimension only:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \quad (1)$$

with Boundary Conditions:

$$T(0,t) = T_0 \quad (2)$$

$$T(\infty,t) = T_b \quad (3)$$

and Initial Condition $T(x,0) = T_b \quad (4)$

Try separation of variables: $T(x,t) = f(x) g(t) \quad (5)$

(5) into (1):

$$\delta f = \alpha f'' g \Rightarrow \frac{\delta f}{f} = \frac{f''}{\alpha g} = -\lambda^2 \quad (6)$$

(6) $\Rightarrow g = Ae^{-\lambda^2 xt}$

$$f = B\cos \lambda x + C\sin \lambda x$$

from (5),

$$T(x,t) = Ae^{-\lambda^2 xt} (B\cos \lambda x + C\sin \lambda x) \quad (7)$$
Can we satisfy all BC's and IC's with (7)?

The answer is no!

\[ T(\infty, t) = A e^{-\alpha t} \left( B \cos \omega t + C \sin \omega t \right) \text{ not defined} \]

Thus, separation of variables does not work!

Let try Laplace Transforms.

**Laplace Transform Review**

Definition:  \[ F(s) = \int_0^\infty e^{-st} f(t) \, dt \]

Some properties:

\[ \mathcal{L} \left[ \frac{\partial f(x,t)}{\partial t} \right] = \int_0^\infty e^{-st} \frac{\partial f}{\partial t} \, dt \rightarrow \text{integration by parts} \]

\[ = f(x,t) e^{-st} \bigg|_{t=0}^{t=\infty} + \int_0^\infty f(x,t) e^{-st} \, dt = \int_0^\infty f(x,s) e^{-st} \, dt \]

\[ = s F(x,s) - f(x,0) \]

\[ \mathcal{L} \left[ \frac{\partial f(x,t)}{\partial x} \right] \text{ if Laplace is done wrt } t \]

\[ = \int_0^\infty e^{-st} \frac{\partial f}{\partial x} \, dt = \int_0^\infty \frac{e^{-st} f(t)}{\partial x} \, dt = \mathcal{L} \left[ \frac{\partial f(x,t)}{\partial x} \right] \]

Since \( s \) is an algebraic variable, \( \frac{\partial F}{\partial x} = \frac{df}{dx} \).
Continuation of Semi-infinite well example

Step 1 - Transform differential equation + B.C.'s

\[ (1) \quad \mathcal{L}\left[ \frac{\partial T}{\partial t} \right] = \mathcal{L}\left[ \alpha \frac{\partial^2 T}{\partial x^2} \right] \]

If \( T(x,0) = \alpha' T \Rightarrow \Delta T(x,0) = T(x,0) = \alpha \frac{\partial^2 T}{\partial x^2} \) \( \quad (8) \)

From (8) and (4) \( \frac{\partial^2 T}{\partial x^2} - \frac{4}{\alpha} T = -\frac{T(x,0)}{\alpha} = -\frac{T_b}{\alpha} \) \( \quad (9) \)

Transform of B.C. eq. (2) \( \mathcal{L}\left[ T(0,t) \right] = T(0,t) = \frac{T_b}{\alpha} \) \( \quad (10) \)

Transform of B.C. eq. (3) \( \mathcal{L}\left[ T(\infty,t) \right] = T(\infty,t) = \frac{T_b}{\alpha} \) \( \quad (11) \)

Equation (9) becomes an ODE since there are only derivatives with respect to \( x \).

Equation (9) is a 2nd-order non-homogeneous ODE.

It's solution is:

\[ T(x,t) = A(x) e^{\sqrt{\frac{2}{\alpha}} x} + B(x) e^{-\sqrt{\frac{2}{\alpha}} x} + \frac{T_b}{\alpha} \]

homogeneous solution

particular solution

Note: \( A \) and \( B \) are independent of \( t \) to satisfy (8) but they may/may not be dependent on \( \Delta \).
Step 2- Satisfy boundary conditions with equation (12)

\[ T(0,\lambda) = \frac{T_0}{\lambda} = A(\lambda) + B(\lambda) + \frac{T_b}{\lambda} \]  

(13)

\[ T(\infty, \lambda) = \frac{T_b}{\lambda} = A(\lambda) e^{-\frac{\sqrt{2}}{\lambda x}} + 0 + \frac{T_b}{\lambda} \]  

(14)

\[ (14) \Rightarrow A(\lambda) = 0 \]  

(15)

\[ (13) \Rightarrow B(\lambda) = \frac{T_0 - T_b}{\lambda} \]  

(16)

\[ \Rightarrow T(x, \lambda) = \frac{T_0 - T_b}{\lambda} e^{-\frac{\sqrt{2}}{\lambda x}} + \frac{T_b}{\lambda} \]  

(17)

Step 3- Inverse Laplace (from Tables/Residues)

\[ T(x, t) = \mathcal{L}^{-1}(T(x, \lambda)) = (T_0 - T_b) \text{erf}\left(\frac{x}{2\sqrt{\lambda}t}\right) + T_b \]  

or

\[ \frac{T(x, t)}{T_b} = 1 + \left(\frac{T_b}{T_0} - 1\right) \text{erf}\left(\frac{x}{2\sqrt{\lambda}t}\right) \]

Comments:

1- Solution is not in separable form \( f(x)g(t) \)

2- \( x \) end \( t \) collapse into a single variable \( \frac{x}{2\sqrt{\lambda}t} \)
Example #2: Semi-infinite wall with a heat-flux boundary condition.

Some differential equation \( \frac{dT}{dt} = \alpha \frac{d^2T}{dx^2} \) \( (1) \)

Initial Condition \( T(x, 0) = T_b \) \( (2) \)

Boundary Conditions \( \psi = -K \frac{dT}{dx} \bigg|_{x=0} \) \( (3) \)

\( T(\infty, t) = T_b \) \( (4) \)

Step 1 - \( \Delta T(x, t) = T(x, 0) = \alpha \frac{d^2T}{dx^2} \) \( (5) \)

(solve Laplace transform of (3)) \( \Rightarrow K \frac{dT}{dx} = \frac{Q}{\lambda} \) \( (6) \)

The solution of (5) is \( \tilde{T}(x, s) = A(s)e^{-\frac{\sqrt{s}}{\lambda} x} + B(s)e^{\frac{\sqrt{s}}{\lambda} x} + \frac{T_b}{\lambda} \) \( (7) \)

Step 2 - Satisfy Boundary Conditions with (7)

For \( x \to \infty \), equation (4) \( \Rightarrow B e^{\frac{\sqrt{s}}{\lambda} \infty} + \frac{T_b}{\lambda} = \frac{T_b}{\lambda} \Rightarrow B = 0 \) \( (8) \)

For \( x = 0 \) \( \frac{dT}{dx} \bigg|_{x=0} = \frac{Q}{\lambda} = -\frac{\sqrt{s}}{\lambda} A \Rightarrow A(s) = \frac{Q\sqrt{s}}{K \lambda^{3/2}} \) \( (9) \)

From (8) and (9) into (7)

\( \tilde{T}(x, s) = \frac{Q}{K} \frac{\sqrt{s}}{\lambda \lambda^{3/2}} e^{-\frac{\sqrt{s}}{\lambda} x} + \frac{T_b}{\lambda} \) \( (10) \)

Step 3 - Inversion of (10) using Transform Table

\( T(x, t) = \frac{Q}{K} \sqrt{\frac{\lambda}{\pi}} \left\{ 2 \sqrt{\frac{t}{\pi}} \exp \left( -\frac{x^2}{4xt} \right) - \frac{x}{\sqrt{\lambda}} \text{erfc} \left( \frac{x}{2\sqrt{xt}} \right) \right\} + T_b \)

Consider a long horizontal pipe of radius \( R \) through which an incompressible fluid is flowing at an average velocity \( V_z \) in the \( z \) direction.

Assume
1. \( u_z \gg V_z \)
2. Conduction in \( r \) larger than in \( z \).
3. Steady State

The energy equation is
\[
\frac{V_z}{c} \frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right) \tag{1}
\]
\[0 \leq z \leq \infty\]
\[0 \leq r \leq R\]

Solve \( T(r,z) \) with Laplace Method. The problem can be also solved by separation of variables.

Boundary Conditions:
\[
T(r,0) = T_0 \quad \text{inlet temperature} \quad \tag{2}
\]
\[
T(R,z) = T_a \quad \text{ambient temperature} \quad \tag{3}
\]

Since the domain is from \( 0 \) to \( \infty \) in \( z \), we will do Laplace transform with respect to the \( z \)-variable.
Step 1 - Transform the equation and boundary conditions

From (1) \[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \frac{v^2}{\alpha} \left[ A T - T(r, \theta) \right] = 0 \] (4)

From (2) and (4) \[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \frac{v^2}{\alpha} \left[ A T - T_0 \right] = 0 \] (5)

From (3) \[ T(r, \theta) = \frac{T_0}{A} \] (6)

After algebra, from (5) \[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \frac{v^2}{\alpha} \left[ A T - \frac{v^2}{\alpha} T_0 \right] = 0 \] (7)

Equation (7) is a nonhomogeneous Modified Bessel Equation:

The solution to the homogeneous part of (7) is:

\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \frac{v^2}{\alpha} A T = 0 \Rightarrow T(r, \theta) = A(\theta) I_0 \left( \sqrt{\frac{4v^2 \alpha}{r}} \right) + B(\theta) K_0 \left( \sqrt{\frac{4v^2 \alpha}{r}} \right) \] (8)

For the particular solution of (7) by substitution:

\[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} - \frac{v^2}{\alpha} A T - \frac{v^2}{\alpha} T_0 = 0 \Rightarrow T = \frac{T_0}{A} \] (9)

Then, the general solution is:

\[ T(r, \theta) = \frac{T_0}{A} + A(\theta) I_0 \left( \sqrt{\frac{4v^2 \alpha}{r}} \right) + B(\theta) K_0 \left( \sqrt{\frac{4v^2 \alpha}{r}} \right) \] (10)

Step 2 - Satisfy boundary conditions with equation (10)

Remember, \( I_0 (r) = X_0 (ir) \) and \( K_0 (r) = Y_0 (ir) \)

From before \( Y_0 (0) \to \infty \Rightarrow K_0 (0) \to 0 \Rightarrow \) from (10) \( B = 0 \)
Then, from (6) and (10) using $\delta = 0$

$$\frac{T_0}{\delta} + A(\delta) \, I_0 \left( \frac{\sqrt{\Delta v^2 \alpha}}{\Delta} R \right) = \frac{T_{A}}{\delta} \Rightarrow A(\delta) = \frac{T_{A} - T_{0}}{\delta \, I_0 \left( \frac{\sqrt{\Delta v^2 \alpha}}{\Delta} R \right)}$$  \hspace{1cm} (11)

Then, from (11) and (10)

$$\hat{T}(r, \delta) = \frac{T_{0}}{\delta} + \frac{T_{A} - T_{0}}{\delta \, I_0 \left( \frac{\sqrt{\Delta v^2 \alpha}}{\Delta} R \right)} \, I_0 \left( \frac{\sqrt{\Delta v^2 \alpha}}{\Delta} r \right)$$  \hspace{1cm} (12)

**Step 3 - Inverse Transform of (12)**

The inverse transform is not available in the Laplace Transform Tables.

Then, we have to use a classic result of complex algebra: Cauchy's Theorem of Residues (Proof. Appendix C Rice/Do book)

See review in the next page.
Review: Inversion Theorem with Pole Singularities

Let \[ F(s) = \int_0^\infty f(t) e^{-st} \, dt \]

Then \[ f(t) = \sum \text{residues of } F(s) e^{st} \]

Residues are computed as follows:

Assume \( s = -a \) is a pole (zero in the denominator of \( F(s) \))

Then \[ \text{res} (s = -a) = \left. \frac{[(s+a) F(s) e^{st}]}{s+a} \right|_{s=-a} \]

Simple Example: Compute \( f(t) \) for:

\[ F(s) = \frac{1}{(s+1)(s+2)} \]

\( F(s) \) has two poles \( s = -1, s = -2 \)

\[ \text{residue } (s = -1) = \left. \frac{1}{(s+1)(s+2)} e^{st} \right|_{s=-1} = e^{-t} \]
\[ \text{residue } (s = -2) = \left. \frac{1}{(s+1)(s+2)} e^{st} \right|_{s=-2} = -e^{-2t} \]

Then, \[ f(t) = \sum \text{residues of } F(s) e^{st} \]
\[ = e^{-t} - e^{-2t} \]

Now, let complete the pipe problem.
$2^{nd}$ Simple Example: Residues for $F(s)$ with repeated roots

$$F(s) = \frac{N(s)}{(s-a)^m}$$

multiple roots at $s = a$ ($m$ roots). What are the residues?

If we expand $N(s)$ with a Taylor Series around $s = a$

$$N(s) = N(a) + (s-a)N'(a) + \frac{(s-a)^2}{2!} N''(a) + \ldots$$

$$+ \ldots \frac{(s-a)^{m-1}}{(m-1)!} N^{(m-1)}(a)$$

Let compute:

$$\text{res}(s=a) = \lim_{s \to a} \left[ (s-a) \frac{N(s) e^{st}}{(s-a)^m} \right] = \frac{N^{(m-1)}(a)}{(m-1)!}$$

$$F(s) = \frac{N(s)}{(s-a)^m} \Rightarrow N(s) = F(s) (s-a)^m$$

$$\Rightarrow N(a) = \frac{d^{m-1}}{ds^{m-1}} \left[ F(s) (s-a)^m \right] \bigg|_{s=a}$$

$$f(t) = \lim_{s \to a} \left[ (s-a) \frac{N(s) e^{st}}{(s-a)^m} \right] = \frac{1}{(m-1)!} \frac{d^{m-1}[e^{st} F(s)(s-a)^m]}{ds^{m-1}}_{s=a}$$

e.g. $\frac{1}{(s-a)^2}$, what is the residue ($s=a$)

$$f(t) = \frac{1}{(2-1)!} t e^{at}$$
Completion of the Long Pipe Problem

Invert Eq (12) \( \Phi (r, \alpha) = \frac{T_0}{\alpha} + \frac{T_0 - T_0}{J_0 (\frac{\sqrt{A}}{\alpha} R)} \quad (12) \)

Since \( T_0 \) is better tabulated than \( J_0 (\frac{\sqrt{A}}{\alpha} r) \), use \( J_0 (\frac{\sqrt{A}}{\alpha} r) = I_0 (\frac{\sqrt{A}}{\alpha} R) \)

Then, define \( i \frac{\sqrt{A}}{\alpha} = 1 \) \( (13) \)

Now, equation (12) becomes:

\[ \Phi (r, \alpha) = \frac{T_0}{\alpha} + \frac{T_0 - T_0}{J_0 (\alpha R)} \]

The poles of (14) are \( \alpha = 0 \) and, from book, the solutions of \( J_0 (\alpha R) = 0 \) \( \alpha_1, \alpha_2, \ldots \)

from (13) \( \Rightarrow \alpha_n = \frac{\alpha_1}{\sqrt{A}} \)

Compute residues

\[
res \left[ \alpha = 0 \right] = \left[ \alpha e^{\frac{4 \pi}{\alpha} \Phi (r, \alpha)} \right]_{\alpha = 0} = \left\{ e^{\frac{4 \pi}{\alpha} \left[ \frac{T_0 + (T_0 - T_0) J_0 (\alpha R)}{J_0 (\alpha R)} \right]} \right\}_{\alpha = 0} = \frac{T_0}{I_0 (\alpha R)}
\]

\[
res \left[ \alpha = \alpha_n \right] = \left[ (\alpha - \alpha_n) e^{\frac{4 \pi}{\alpha} \Phi (r, \alpha)} \right]_{\alpha = \alpha_n} = \left[ (\alpha - \alpha_n) e^{\left[ \frac{T_0 + (T_0 - T_0) J_0 (\alpha R)}{I_0 (\alpha R)} \right]} \right]_{\alpha = \alpha_n}
\]

\[
= \left[ (\alpha - \alpha_n) e^{\frac{T_0}{I_0 (\alpha R)}} \right]_{\alpha = \alpha_n} + \left[ (\alpha - \alpha_n) e^{\frac{T_0}{I_0 (\alpha R)}} \right]_{\alpha = \alpha_n}
\]

However the second term in the last equation \( \frac{0}{0} \), \( r = R \).

Therefore, we need to apply L'Hopital rule to find limit.
\[ (T_q - T_0) \quad \frac{d}{ds} \left\{ e^{\frac{2}{v_s^2} (4 - \lambda_n)} J_0(\lambda_n r) \right\}_{\lambda_n=\lambda_n} \frac{d}{ds} \left[ 1 + J_0(\lambda_n R) \right] \quad \frac{d}{ds} \left[ 1 + J_0(\lambda_n R) \right]_{\lambda_n=\lambda_n} \]

\[ \text{(15)} \]

Using a derivation property of Bessel functions:

from (13)

\[ \frac{d}{ds} \left[ 1 + J_0(\lambda_n R) \right] = J_0(\lambda_n R) - R \frac{\lambda_n^2}{2} J_1(\lambda_n R) \]

\[ = - R \frac{\lambda_n^2}{2} J_1(\lambda_n R) \quad \lambda_n=\lambda_n \]

\[ (16) \]

from (15) and (16)

\[ \Rightarrow \text{res} \left[ \lambda_n = \lambda_n \right] = (T_q - T_0) \quad \frac{d}{ds} \left\{ e^{\frac{2}{v_s^2} (4 - \lambda_n)} J_0(\lambda_n r) \right\}_{\lambda_n=\lambda_n} \frac{d}{ds} \left[ 1 + J_0(\lambda_n R) \right] - \frac{R}{2} \frac{\lambda_n^2}{\lambda_n} J_1(\lambda_n R) \]

\[ = \frac{R}{2} \frac{\lambda_n^2}{\lambda_n} J_1(\lambda_n R) \]

\[ T(r, t) = \sum \text{(residues)} \Rightarrow \]

\[ T(r, t) = T_0 - 2 \frac{(T_q - T_0)}{R} \sum_{\lambda_n} e^{-\frac{\lambda_n^2}{v_s^2} t} \frac{J_0(\lambda_n r)}{J(\lambda_n R)} \]

For \( t \to \infty \to e^{-\frac{\lambda_n^2}{v_s^2} t} = 0 \) \( \Rightarrow T(r, t) = T_0 \)

It's correct, at the end of the long pipe the liquid exits at ambient temperature.