Solution of PDE's using Laplace

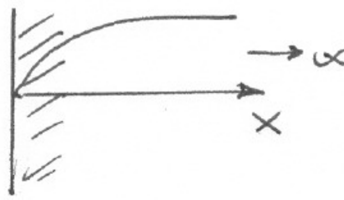
When?

- Problem cannot be solved with seperation of veriebles
- Transform with respect to vericble that goes from $0 \to \infty$
  e.g. time $(0 < t < \infty)$

Motivation Example

Solve the temperature $T(x,t)$ in a semi-infinite well (very thick well)

Heat transfer in 1 dimension only:



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad (1)$$

with Boundery Conditions:

$$T(0,t) = T_0 \qquad (2)$$

$$T(\infty,t) = T_b \qquad (3)$$

and Initial Condition $T(x,0) = T_b$ (4)

Try seperation of veriebles: $T(x,t) = f(x)\,g(t)$ (5)

(5) into (1):

$$g'f = \alpha f''g \Rightarrow \frac{g'}{\alpha g} = \frac{f''}{f} = -\lambda^2 \qquad (6)$$

$$(6) \Rightarrow \quad g = Ae^{-\lambda^2 \alpha t} \quad \text{and} \quad f = B\cos\lambda x + C\sin\lambda x$$

$\Rightarrow$ from (5)

$$T(x,t) = Ae^{-\lambda^2\alpha t}(B\cos\lambda x + C\sin\lambda x) \qquad (7)$$

Can we satisfy all BC's and IC's with (7) ?

The answer is no!

$$T(\infty,t) = Ae^{-\lambda^2 \alpha t}\left(B\cos\lambda\infty + C\sin\lambda\infty\right) \text{ not defined}$$

Thus, separation of variables does not work!

Let try Laplace Transforms.

<u>Laplace Transform Review</u>

Definition $\quad F(s) = \int_0^\infty e^{-st} f(t)\, dt$

Some properties

$$\mathcal{L}\left[\frac{\partial f(x,t)}{\partial t}\right] = \int_0^\infty e^{-st}\frac{\partial f}{\partial t}\, dt \rightarrow \text{integration by parts}$$

$$= f(x,t)\, e^{-st}\Big|_{t=0}^{t=\infty} + s\int_0^\infty f(x,t) e^{-st} dt =$$

$$= sF(x,s) - f(x,0)$$

$\mathcal{L}\left[\frac{\partial f}{\partial x}(x,t)\right]$? if Laplace is done wrt t

$$= \int_0^\infty e^{-st}\frac{\partial f}{\partial x}\, dt = \frac{\partial}{\partial x}\int_0^\infty e^{-st} f(t)\, dt = \frac{\partial [F(x,s)]}{\partial x}$$

since s is an algebraic variable $\frac{\partial F}{\partial x} = \frac{dF}{dx}$

Continuation of Semi-infinite wall example

Step 1 - Transform differential equation + B.C.'s

eq. (1) $$\mathcal{L}\left[\frac{\partial T}{\partial t}\right] = \mathcal{L}\left[\alpha \frac{\partial^2 T}{\partial x^2}\right]$$

if $\bar{T}(x,s) = \mathcal{L}(T)$ $\Rightarrow$ $$s\bar{T}(x,s) - T(x,0) = \alpha \frac{\partial^2 \bar{T}}{\partial x^2} \quad (8)$$

From (8) and (4) $$\frac{\partial^2 \bar{T}}{\partial x^2} - \frac{s}{\alpha}\bar{T} = -\frac{T(x,0)}{\alpha} = -\frac{T_b}{\alpha} \quad (9)$$

Transform of B.C. eq. (2) $\mathcal{L}\left[T(0,t)\right] = \bar{T}(0,s) = \frac{T_0}{s}$ (10)

Transform of B.C. eq (3) $\mathcal{L}\left[T(\infty,t)\right] = \bar{T}(\infty,s) = \frac{T_b}{s}$ (11)

Equation (9) becomes an ODE since there are only derivatives with respect to x

Equation (9) is a 2^{nd}-order non-homogeneous ODE.

It's solution is:

$$\bar{T}(x,s) = \underbrace{A(s)\, e^{\sqrt{\frac{s}{\alpha}}x} + B(s)\, e^{-\sqrt{\frac{s}{\alpha}}x}}_{\text{homogeneous solution}} + \underbrace{\frac{T_b}{s}}_{\text{particular solution}} \quad (12)$$

Note: A and B are independent of t to satisfy (9) but they may/may not be dependent on s.

Step 2- Satisfy boundary conditions with equation (12)

$$\bar{T}(0,s) = \frac{T_o}{s} = A(s) + B(s) + \frac{T_b}{s} \qquad (13)$$

$$\bar{T}(\infty,s) = \frac{T_b}{s} = A(s)\, e^{\sqrt{\frac{s}{\alpha}}\infty} + 0 + \frac{T_b}{s} \qquad (14)$$

$$(14) \Rightarrow A(s) = 0 \qquad (15)$$

$$(13) \Rightarrow B(s) = \frac{T_o - T_b}{s} \qquad (16)$$

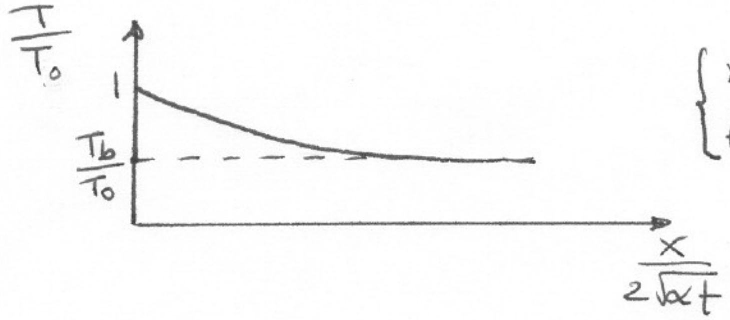
$$\Rightarrow \bar{T}(x,s) = \frac{T_o - T_b}{s}\, e^{-\sqrt{\frac{s}{\alpha}}\,x} + \frac{T_b}{s} \qquad (17)$$

Step 3- Inverse Laplace (from Tables / Residues)

$$T(x,t) = \mathcal{L}^{-1}\left(\bar{T}(x,s)\right) = (T_o - T_b)\, \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + T_b$$

or

$$\frac{T(x,t)}{T_b} = 1 + \left(\frac{T_b}{T_o} - 1\right) \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$



$$\begin{cases} x \to \infty & T = T_b \text{ for all } t \\ t \to \infty & T = T_o \text{ for all } x \end{cases}$$

Comments:

1- Solution is not in separable form $(f(x)\,g(t))$

2- x and t collapse into a single variable $x/2\sqrt{\alpha t}$

Example #2: Semi-infinite wall with a heat-flux boundary condition.

Same differential equation $\quad \dfrac{\partial T}{\partial t} = \alpha \dfrac{\partial^2 T}{\partial x^2} \qquad (1)$

Initial Condition $\quad T(x,0) = T_b \qquad (2)$

Boundary Conditions $\quad q = -k \left.\dfrac{\partial T}{\partial x}\right|_{x=0} \qquad (3)$

$$T(\infty,t) = T_b \qquad (4)$$

Step 1- $\quad s\bar{T}(x,s) - T(x,0) = \alpha \dfrac{\partial^2 \bar{T}}{\partial x^2} \qquad (5)$

Laplace transform of (3) $\rightarrow k \dfrac{\partial \bar{T}}{\partial x} = \dfrac{q}{s} \qquad (6)$

The solution of (5) is $\quad \bar{T}(x,s) = A(s)e^{-\sqrt{\frac{s}{\alpha}}x} + B(s)e^{\sqrt{\frac{s}{\alpha}}x} + \dfrac{T_b}{s} \quad (7)$

Step 2- Satisfy Boundary Conditions with (7)

For $x \to \infty$, equation (4) $\Rightarrow Be^{\sqrt{\frac{s}{\alpha}}\infty} + \dfrac{T_b}{s} = \dfrac{T_b}{s} \Rightarrow B = 0 \quad (8)$

For $x = 0 \quad \left.\dfrac{\partial \bar{T}}{\partial x}\right|_{x=0} = \dfrac{q}{s} = -\sqrt{\dfrac{s}{\alpha}}\,A \Rightarrow A(s) = \dfrac{q\sqrt{\alpha}}{k\, s^{3/2}} \quad (9)$

From (8) and (9) into (7)

$$\bar{T}(x,s) = \frac{q\sqrt{\alpha}}{k} \frac{e^{-\sqrt{\frac{s}{\alpha}}x}}{s^{3/2}} + \frac{T_b}{s} \qquad (10)$$

Step 3- Inversion of (10) using Transform Table

$$T(x,t) = \frac{q\sqrt{\alpha}}{k} \left\{ 2\sqrt{\frac{t}{\pi}} \exp\left(-\frac{x^2}{4\alpha t}\right) - \frac{x}{\sqrt{\alpha}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \right\} + T_b$$

Example 3: Steady State Conduction/Convection in a Long Pipe.

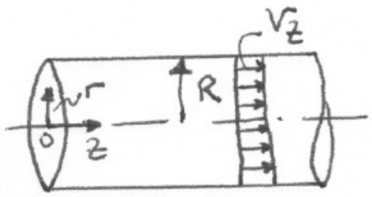
Consider a long horizontal pipe of radius R through which an incompressible fluid is flowing at an average velocity v_z in the z direction.

Assume

1- $v_z >> v_r$

2- conduction in r larger than in z.

3- Steady state



The energy equation is
$$v_z \frac{\partial T}{\partial z} = \alpha \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} \right) \qquad (1)$$

$$0 \le z \le \infty$$
$$0 \le r \le R$$

Solve $T(r,z)$ with Laplace Method. The problem can be also solved by separation of variables.

Boundary Conditions:

$$T(r,0) = T_0 \qquad \text{inlet temperature} \qquad (2)$$
$$T(R,z) = T_A \qquad \text{ambient temperature} \qquad (3)$$

Since the domain is from 0 to ∞ in z, we will do Laplace transform with respect to the z-variable.

Step 1- Transform the equation and boundary conditions

From (1) $$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{V_z}{\alpha}\left[s\bar{T} - T(r,0)\right] = 0 \quad (4)$$

From (2) and (4) $$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{V_z}{\alpha}\left[s\bar{T} - T_0\right] = 0 \quad (5)$$

From (3) $$\bar{T}(R,s) = \frac{T_A}{s} \quad (6)$$

After algebra, from (5) $$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{V_z}{\alpha} s\bar{T} = \frac{V_z T_0}{\alpha} \quad (7)$$

Equation (7) is a nonhomogeneous Modified Bessel Equation:

The solution to the homogeneous part of (7) is:

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{V_z}{\alpha} s\bar{T} = 0 \Rightarrow \bar{T}(r,s) = A(s)\, I_0\left(\sqrt{\frac{sV_z}{\alpha}}\, r\right) + B(s)\, K_0\left(\sqrt{\frac{sV_z}{\alpha}}\, r\right) \quad (8)$$

For the particular solution of (7), by substitution:

$$\frac{\partial^2 \bar{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{T}}{\partial r} - \frac{V_z}{\alpha} s\bar{T} = \frac{V_z T_0}{\alpha} \rightarrow \bar{T} = \frac{1}{s} \cdot T_0 \quad (9)$$

Then, the general solution is:

$$\bar{T}(r,s) = \frac{T_0}{s} + A(s)\, I_0\left(\sqrt{\frac{sV_z}{\alpha}}\, r\right) + B(s)\, K_0\left(\sqrt{\frac{sV_z}{\alpha}}\, r\right) \quad (10)$$

Step 2- Satisfy boundary conditions with equation (10)

Remember, $I_0(r) = J_0(ir)$ and $K_0(r) = Y_0(ir)$

From before $Y_0(0) \rightarrow \infty \Rightarrow K_0(0) \rightarrow 0 \Rightarrow$ from (10) $B = 0$

Then, from (6) and (10) using $B=0$

$$\frac{T_o}{s} + A(s)\, I_o\left(\sqrt{\frac{sV_z}{\alpha}}\,R\right) = \frac{T_A}{s} \Rightarrow A(s) = \frac{T_A - T_o}{s\, I_o\left(\sqrt{\frac{sV_z}{\alpha}}\,R\right)} \qquad (11)$$

Then, from (11) and (10)

$$\bar{T}(r,s) = \frac{T_o}{s} + \frac{T_A - T_o}{s\, I_o\left(\sqrt{\frac{sV_z}{\alpha}}\,R\right)}\, I_o\left(\sqrt{\frac{sV_z}{\alpha}}\,r\right) \qquad (12)$$

Step 3- Inverse Transform of (12)

The inverse transform is not available in the Laplace Transform Tables.

Then, we have to use a classic result of complex algebra: <u>Cauchy's Theorem of Residues</u> (Proof: Appendix C Rice/Do book)

See review in the next page.

Review: Inversion Theorem with Pole Singularities

Let
$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

Then
$$f(t) = \Sigma\,(\text{residues of } F(s)\, e^{st})$$

Residues are computed as follows:

Assume $s = -a$ is a pole (zero in the denominator of $F(s)$)

Then
$$\text{res}\,(s = -a) = \left[(s+a)\, F(s)\, e^{st}\right]\Big|_{s=-a}$$

Simple Example: compute $f(t)$ for:

$$F(s) = \frac{1}{(s+1)\,(s+2)}$$

$F(s)$ has two poles $s = -1$ $s = -2$

$$\text{residue}\,(s=-1) = \left[\cancel{(s+1)}\, \frac{1}{\cancel{(s+1)}(s+2)}\, e^{st}\right]\Big|_{s=-1} = e^{-t}$$

$$\text{residue}\,(s=-2) = \left[(s+2)\, \frac{1}{(s+1)\,(s+2)}\, e^{st}\right]\Big|_{s=-2} = -e^{-2t}$$

Then,
$$f(t) = \Sigma\,(\text{residues of } F(s)\, e^{st})$$
$$= e^{-t} - e^{-2t}$$

Now, let complete the pipe problem.

2nd Simple Example: Residues for $F(s)$ with repeated roots

$$F(s) = \frac{N(s)}{(s-a)^m} \qquad \cancel{\left[(s-a)\frac{N(s)}{(s-a)^m} \cdot e^{st}\right]\Big|_{s=a}} \to \infty$$

multiple roots at $s=a$ (m roots). What are the residues?

If we expand $N(s)$ with a Taylor Series around $s=a$

$$N(s) = N(a) + (s-a)N'(a) + \frac{(s-a)^2}{2!}N''(a) + \cdots + \cdots \frac{(s-a)^{m-1}}{(m-1)!}N^{(m-1)}(a)$$

Let compute:

$$res(s=a) = \lim_{s\to a}\left[(s-a)\frac{N(s)}{(s-a)^m}e^{st}\right] = \frac{N^{(m-1)}(a)}{(m-1)!}$$

$$F(s) = \frac{N(s)}{(s-a)^m} \Rightarrow N(s) = F(s)(s-a)^m$$

$$\Rightarrow N^{(m-1)}(a) = \left\{\frac{d^{m-1}}{ds^{m-1}}\left[F(s)(s-a)^m\right]\right\}\Big|_{s=a}$$

$$f(t) = \lim\left[(s-a)\frac{N(s)}{(s-a)^m}e^{st}\right] = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}\left[e^{st}F(s)(s-a)^m\right]}{ds^{m-1}}\right\}\Big|_{s=a}$$

e.g. $\frac{1}{(s-a)^2}$, what is the residue ($s=a$)

$$f(t) = \frac{1}{(2-1)!}\, t e^{at}$$

Completion of the Long Pipe Problem

Invert Eq (12) $\rightarrow \bar{T}(r,s) = \dfrac{T_0}{s} + \dfrac{T_A - T_0}{s\, I_0\left(\sqrt{\frac{s v_z}{\alpha}}\, R\right)}\, I_0\left(\sqrt{\frac{s v_z}{\alpha}}\, r\right)$ (12)

Since J_0 is better tabulated then I_0 : use $J_0\left(i\sqrt{\frac{s v_z}{\alpha}}\, r\right) = I_0\left(\sqrt{\frac{s v_z}{\alpha}}\, r\right)$

Then, Define $i\sqrt{\dfrac{s v_z}{\alpha}} = \lambda$ (13)

Now, equation (12) becomes:

$$\bar{T}(r,s) = \frac{T_0}{s} + \frac{T_A - T_0}{J_0(\lambda R)}\, J_0(\lambda r) \qquad (14)$$

The poles of (14) are $s = 0$ and,
from book, the solutions of $J_0(\lambda R) = 0$ λ_n; $n = 1, 2, \ldots$

from (13) $\Rightarrow s_n = -\lambda_n^2 \dfrac{\alpha}{v_z}$ (14.a)

Compute residues

$$\mathrm{res}[s=0] = \left[s e^{sz} \bar{T}(r,s)\right]\Big|_{s=0} = \left\{ e^{sz}\left[T_0 + (T_A - T_0)\frac{\overset{J_0(0)=1}{J_0(\lambda_n r)}}{J_0(\lambda_n R)}\right]\right\}_{s=0} = T_A$$

$$\mathrm{res}[s = s_n] = \left[(s - s_n) e^{sz} \bar{T}(r,s)\right]\Big|_{s=s_n} = \left[(s-s_n) e^{sz}\left(\frac{T_0}{s} + (T_A - T_0)\frac{J_0(\lambda_n r)}{J_0(\lambda_n R)}\right)\right]_{s=s_n} =$$

$$= \underbrace{\left[(s-s_n) e^{sz}\frac{T_0}{s}\right]_{s=s_n}}_{=0} + \left[(s-s_n) e^{sz}(T_A - T_0)\frac{J_0(\lambda_n r)}{J_0(\lambda_n R)}\right]_{s=s_n}$$

However the second term in the last equation $= \dfrac{0}{0}$ $(r=R)$
Therefore, we need to apply L'Hopital rule to find limit.

$$= (T_A - T_0)\,\frac{\frac{d}{ds}\left\{e^{sz}(s-\lambda_n)\,J_0(\lambda_n r)\right\}_{s=\lambda_n}}{\frac{d}{ds}\left[s\,J_0(\lambda_n R)\right]_{s=\lambda_n}} = \frac{(T_A - T_0)\,e^{\lambda_n z}\,J_0(\lambda_n r)}{\frac{d}{ds}\left[s\,J_0(\lambda_n R)\right]_{s=\lambda_n}} \qquad (15)$$

Using a derivation property of Bessel functions:

from (13) $$\frac{d}{ds}\left[s\,J_0\left(i\sqrt{\frac{s v_z}{\alpha}}R\right)\right] = \left\{J_0\left(i\sqrt{\frac{s v_z}{\alpha}}R\right) - \frac{iR}{2}\sqrt{\frac{s v_z}{\alpha}}\,J_1\left(i\sqrt{\frac{s v_z}{\alpha}}R\right)\right\}_{s=\lambda_n}$$

$$= -\frac{R}{2}\lambda_n\,J_1(\lambda_n R) \qquad (16)$$

from (15) and (16)

$$\Rightarrow \text{res}\,[s=\lambda_n] = (T_A - T_0)\,\frac{e^{\lambda_n z}\,J_0(\lambda_n r)}{-\frac{R}{2}\lambda_n\,J_1(\lambda_n R)} = -\frac{2(T_A - T_0)}{R\lambda_n}\,\frac{e^{-\lambda_n^2\frac{\alpha}{v_z}z}\,J_0(\lambda_n r)}{J_1(\lambda_n R)}$$

$$T(r,t) = \Sigma(\text{residues}) \Rightarrow$$

$$T(r,t) = T_A - \frac{2(T_A - T_0)}{R\lambda_n}\sum_{n=0}^{\infty} e^{-\lambda_n^2\frac{\alpha}{v_z}z}\,\frac{J_0(\lambda_n r)}{J_1(\lambda_n R)}$$

For $z \to \infty \to e^{-\lambda_n^2\frac{\alpha}{v_z}z} = 0 \quad \Rightarrow T(r,t) = T_A$

It's correct, at the end of the long pipe the liquid exits at ambient temperature.