**Structured Singular Valued based Robust Nonlinear Model Predictive Controller using Volterra series models**

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**Supporting Information**

The interconnection matrix $\mathbf{M}$ is as follows:

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{21} & \mathbf{M}_{22} \end{array}\right] = \left[\begin{array}{c|c} \mathbf{M}_{\mathrm{A}} & \mathbf{M}_{\mathrm{B}} \\ \hline \mathbf{M}_{\mathrm{C}} & \mathbf{M}_{\mathrm{D}} \end{array}\right] \tag{1}$$

the matrices $\mathbf{M}_{\mathrm{A}}$ and $\mathbf{M}_{\mathrm{D}}$ are matrices of appropriate dimensions with all elements equal to zero. The notation $\mathbf{0}_{(i\times j)}$ refers heretofore to a matrix of $i$ rows and $j$ columns that have all of its elements equal to zero. If $i$ and $j$ are not specified then $\mathbf{0}$ refers to a matrix of appropriate dimensions that have all of its elements equal to zero. $\mathbf{M}_{\mathrm{B}}$ is constructed as follows:

$$\mathbf{M}_{\mathrm{B}} = \left[\mathbf{M}_{\mathrm{B1}}, \mathbf{M}_{\mathrm{B2}}\right]^{\mathrm{T}} \tag{2}$$

$$\mathbf{M}_{\mathrm{B1}} = \mathrm{diag}\left[\mathbf{M}_{\mathrm{B1A}}, \mathbf{M}_{\mathrm{B1B}}\right] \tag{3}$$

$$\mathbf{M}_{\mathrm{B1A}} = k_{ssv}\mathrm{diag}\left[(\mathbf{I}_p)_1, \ldots, (\mathbf{I}_p)_{n_y}\right] \tag{4}$$

$$\mathbf{M}_{\mathrm{B1B}} = k_{ssv}\mathrm{diag}\left[\mathbf{M}_{\mathrm{B1BA}}, \mathbf{M}_{\mathrm{B1BB}}\right] \tag{5}$$

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$$\mathbf{M}_{\mathrm{B1BA}} = \left[\mathbf{M}_{\mathrm{B1BA},1}, \ldots, \mathbf{M}_{\mathrm{B1BA},n_u}\right] \tag{6}$$

$$\mathbf{M}_{\mathrm{B1BA},q} = \operatorname{diag}\left[\left[\begin{array}{c} u_q\left(k-1\right) \\ u_q\left(k\right) \end{array}\right], \ldots, \left[\begin{array}{c} u_q\left(k+m-1\right) \\ u_q\left(k+m\right) \end{array}\right]\right]_{q=1,\ldots,n_u} \tag{7}$$

$$\mathbf{B}_{\mathrm{B1BB}} = \left[\operatorname{diag}\left[u_1\left(k\right), \ldots, u_1\left(k+m\right)\right], \ldots, \operatorname{diag}\left[u_{n_u}\left(k\right), \ldots, u_{n_u}\left(k+m\right)\right]\right] \tag{8}$$

$$\mathbf{M}_{\mathrm{B2}} = k_{ssv}\left[\left[\begin{array}{c} \mathbf{U}^{\mathrm{L}} \\ \mathbf{U}^{\mathrm{NL}} \end{array}\right], \mathbf{0}\right] \tag{9}$$

$$\mathbf{U}^{\mathrm{L}} = \left[\begin{array}{c} \left[\begin{array}{c} \mathbf{U}_1^{\mathrm{L1}} \\ \mathbf{U}_1^{\mathrm{L2}} \end{array}\right] \\ \vdots \\ \left[\begin{array}{c} \mathbf{U}_p^{\mathrm{L1}} \\ \mathbf{U}_p^{\mathrm{L2}} \end{array}\right] \end{array}\right] \tag{10}$$

$$\begin{aligned} \mathbf{U}_i^{\mathrm{L1}} =& \operatorname{diag}\Big[\left[\mathbf{0}_{((p+1-i)\times(i-1))}, u_1\left(i\right)\mathbf{I}_{p+1-i}\right], \ldots, \\ & \left[\mathbf{0}_{((p+1-i)\times((p\times(n_u-1))+(i-1)))}, u_{n_u}\left(i\right)\mathbf{I}_{p+1-i}\right]\Big]_{i=1,\ldots,p} \end{aligned} \tag{11}$$

$$\begin{aligned} \mathbf{U}_i^{\mathrm{L2}} =& \operatorname{diag}\Big[\left[\mathbf{0}_{((p+1-i)\times(i-1))}, \left(u_1\left(i\right)\right)^2\mathbf{I}_{p+1-i}\right], \ldots, \\ & \left[\mathbf{0}_{((p+1-i)\times((p\times(n_u-1))+(i-1)))}, \left(u_{n_u}\left(i\right)\right)^2\mathbf{I}_{p+1-i}\right]\Big]_{i=1,\ldots,p} \end{aligned} \tag{12}$$

$$\mathbf{U}^{\mathrm{NL}} = \left[\mathbf{U}_{1,1}^{\mathrm{NL}}, \mathbf{U}_{1,2}^{\mathrm{NL}}, \ldots, \mathbf{U}_{ii,jj-1}^{\mathrm{NL}}, \mathbf{U}_{ii,jj}^{\mathrm{NL}}\right]_{ii=1,\ldots,p-1;\, jj=1,\ldots,p-ii} \tag{13}$$

$$\mathbf{U}_{ii,jj}^{\mathrm{NL}} = \mathrm{diag}\left[\mathbf{U}_{ii,jj,1}^{\mathrm{NL}}, \ldots, \mathbf{U}_{ii,jj,n_u}^{\mathrm{NL}}\right]_{ii=1,\ldots,p;\, jj=1,\ldots,p-ii} \tag{14}$$

$$\begin{aligned} \mathbf{U}_{ii,jj,q}^{\mathrm{NL}} = \big[&\mathbf{0}_{((p+1-ii-jj)\times(p\times(n_u-1)+(ii+jj-1)))}, \\ &u_q\,(ii+jj)\,u_q\,(jj)\,\mathbf{I}_{p+1-ii-jj}\big]_{ii=1,\ldots,p;\, jj=1,\ldots,p;\, q=1,\ldots,n_u} \end{aligned} \tag{15}$$

$\mathbf{M}_{\mathrm{C}}$ is constructed according to the following equations:

$$\mathbf{M}_{\mathrm{C}} = \begin{bmatrix} \mathbf{M}_{\mathrm{C1}} \\ \mathbf{M}_{\mathrm{C2}} \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \mathbf{M}_{\mathrm{C1}} \\ \mathbf{M}_{\mathrm{C2}} \end{bmatrix} = \begin{bmatrix} \mathbf{M}_{\mathrm{C1A}}, \mathbf{M}_{\mathrm{C1B}}, \mathbf{M}_{\mathrm{C1C}}, \mathbf{M}_{\mathrm{C1D}}, \mathbf{M}_{\mathrm{C1E}} \\ \mathbf{M}_{\mathrm{C2A}}, \mathbf{M}_{\mathrm{C2B}}, \mathbf{M}_{\mathrm{C2C}}, \mathbf{M}_{\mathrm{C2D}}, \mathbf{M}_{\mathrm{C2E}} \end{bmatrix} \tag{17}$$

$$\mathbf{M}_{\mathrm{C2A}} = \begin{bmatrix} [\mathbf{M}_{\mathrm{C2AA1}}, \mathbf{M}_{\mathrm{C2AA2}}] & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{\mathrm{C2AB}} \end{bmatrix} \tag{18}$$

$$\mathbf{M}_{\mathrm{C2AA1}} = \mathrm{diag}\left[d_1\mathbf{I}_p, \ldots, d_{n_y}\mathbf{I}_p\right] \tag{19}$$

$$\mathbf{M}_{\mathrm{C2AA2}} = \mathrm{diag}\left[ic_1\mathbf{I}_p, \ldots, ic_{n_y}\mathbf{I}_p\right] \tag{20}$$

$$\mathbf{M}_{\mathrm{C2AB}} = \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2AB1}}, \mathbf{M}_{\mathrm{C2AB2}}\right] \tag{21}$$

$$\mathbf{M}_{\mathrm{C2AB1}} = \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2AB1},1}, \ldots, \mathbf{M}_{\mathrm{C2AB1},n_u}\right] \tag{22}$$

$$\mathbf{M}_{\mathrm{C2AB1},q} = \mathrm{diag}\left[ \left[W_1^{\Delta u,q}, -W_1^{\Delta u,q}\right], \ldots, \left[W_m^{\Delta u,q}, -W_m^{\Delta u,q}\right] \right]_{q=1,\ldots,n_u} \tag{23}$$

$$\mathbf{M}_{\mathrm{C2AB2}} = \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2AB2},1}, \ldots, \mathbf{M}_{\mathrm{C2AB2},n_u}\right] \tag{24}$$

$$\mathbf{M}_{\mathrm{C2AB2},q} = \mathbf{I}_m \left( \frac{k_{ssv}}{u_q^{\mathrm{bound}}} \right)_{q=1,\ldots,n_u} \tag{25}$$

$$\mathbf{M}_{\mathrm{C2B}} = \begin{bmatrix} \left[\mathbf{M}_{\mathrm{C2B},1}, \ldots, \mathbf{M}_{\mathrm{C2B},n_u}\right] \\ 0 \end{bmatrix} \tag{26}$$

$$\mathbf{M}_{\mathrm{C2B},q} = \begin{bmatrix} \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2BL},1,1,q}, \ldots, \mathbf{M}_{\mathrm{C2BL},1,n_y,q}\right], \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2BNL},1,1,q}, \ldots, \mathbf{M}_{\mathrm{C2BNL},1,n_y,q}\right] \\ \vdots \\ \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2BL},p,1,q}, \ldots, \mathbf{M}_{\mathrm{C2BL},p,n_y,q}\right], \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2BNL},p,1,q}, \ldots, \mathbf{M}_{\mathrm{C2BNL},p,n_y,q}\right] \end{bmatrix}^{\mathrm{T}}_{q=1,\ldots,n_u} \tag{27}$$

if $(p-1) \geq M$, then $\mathbf{M}_{\mathrm{C2BL}}$, $\mathbf{M}_{\mathrm{C2BNL}}$, $\mathbf{H}_{\mathrm{L1}}$ and $\mathbf{H}_{\mathrm{L2}}$ are calculated as follows:

$$\mathbf{M}_{\mathrm{C2BL},i,j,q} = \begin{bmatrix} \mathbf{0}_{((i-1)\times M)} & \mathbf{0}_{((i-1)\times 1)} \\ \mathbf{H}_{\mathrm{L1},i,j,q} & \mathbf{0}_{(M\times 1)} \\ \mathbf{0}_{((M-(i-1))\times M)} & \mathbf{0}_{((M-(i-1))\times 1)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{28}$$

$$\mathbf{M}_{\mathrm{C2BNL},i,j,q} = \begin{bmatrix} \mathbf{0}_{((i-1)\times M)} & \mathbf{0}_{((i-1)\times 1)} \\ \mathbf{H}_{\mathrm{L2},i,j,q} & \mathbf{0}_{(M\times 1)} \\ \mathbf{0}_{((M-(i-1))\times M)} & \mathbf{0}_{((M-(i-1))\times 1)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{29}$$

$\mathbf{H}_{\mathrm{L1},i,j,q}$ is calculated as follows, if $i < M$

$$\mathbf{H}_{\mathrm{L}1,1,j,q} = \operatorname{diag}\begin{bmatrix} \left[h^{\mathrm{NL}}_{(1,M)\,(j,q)} u_q\left(-M+2\right), \ldots, h^{\mathrm{NL}}_{(1,2)\,(j,q)} u_q\left(0\right), h^{\mathrm{L}}_{(1)\,(j,q)}\right] \\ \vdots \\ \left[h^{\mathrm{NL}}_{(M-1,M)\,(j,q)} u_q\left(0\right), h^{\mathrm{L}}_{(M-1)\,(j,q)}\right] \\ h^{\mathrm{L}}_{(M)\,(j,q)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{30}$$

$$\mathbf{H}_{\mathrm{L}1,2,j,q} = \operatorname{diag}\begin{bmatrix} \left[h^{\mathrm{NL}}_{(1,M)\,(j,q)} u_q\left(-M+3\right), \ldots, h^{\mathrm{NL}}_{(1,3)\,(j,q)} u_q\left(0\right), h^{\mathrm{L}}_{(1)\,(j,q)}\right] \\ \vdots \\ \left[h^{\mathrm{NL}}_{(M-2,M)\,(j,q)} u_q\left(0\right), h^{\mathrm{L}}_{(M-2)\,(j,q)}\right] \\ h^{\mathrm{L}}_{(M-1)\,(j,q)} \\ h^{\mathrm{L}}_{(M)\,(j,q)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{31}$$

$$\mathbf{H}_{\mathrm{L}1,M-1,j,q} = \operatorname{diag}\left[\,\left[h^{\mathrm{NL}}_{(1,M)\,(j,q)} u_q\left(0\right), h^{\mathrm{L}}_{(1)\,(j,q)}\right], h^{\mathrm{L}}_{(2)\,(j,q)}, \ldots, h^{\mathrm{L}}_{(M)\,(j,q)}\right]_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{32}$$

if $i \geq M$

$$\mathbf{H}_{\mathrm{L}1,i,j,q} = \operatorname{diag}\left[h^{\mathrm{L}}_{(1)\,(j,q)}, \ldots, h^{\mathrm{L}}_{(M)\,(j,q)}\right]_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{33}$$

$$\mathbf{H}_{\mathrm{L}2,i,j,q} = \operatorname{diag}\left[h^{\mathrm{NL}}_{(1,1)\,(j,q)}, h^{\mathrm{NL}}_{(2,2)\,(j,q)}, \ldots, h^{\mathrm{NL}}_{(M-1,M-1)\,(j,q)}, h^{\mathrm{NL}}_{(M,M)\,(j,q)}\right]_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{34}$$

If $(p-1) < M$, then $\mathbf{M}_{\mathrm{C2BL}}$, $\mathbf{M}_{\mathrm{C2BNL}}$, $\mathbf{H}_{\mathrm{L1}}$ and $\mathbf{H}_{\mathrm{L2}}$ are calculated as follows:

$$\mathbf{M}_{\mathrm{C2BL},i,j,q} = \begin{bmatrix} \mathbf{0}_{(i-1)\times(p+1-i)} & \mathbf{0}_{((ii-1)\times 1)} \\ \mathbf{H}_{\mathrm{L}1,i,j,q} & \mathbf{0}_{((p+1-i)\times 1)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{35}$$

$$\mathbf{M}_{\mathrm{C2BNL},i,j,q} = \begin{bmatrix} \mathbf{0}_{(i-1)\times(p+1-i)} & \mathbf{0}_{((ii-1)\times 1)} \\ \mathbf{H}_{\mathrm{L2},i,j,q} & \mathbf{0}_{((p+1-i)\times 1)} \end{bmatrix}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{36}$$

$$\mathbf{H}_{\mathrm{L1},i,j,q} = \mathrm{diag}\left[h^{\mathrm{L}}_{(1)\,(j,q)}, \ldots, h^{\mathrm{L}}_{(p+1-i)\,(j,q)}\right]_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{37}$$

$$\mathbf{H}_{\mathrm{L2},i,j,q} = \mathrm{diag}\left[h^{\mathrm{NL}}_{(1,1)\,(j,q)}, h^{\mathrm{NL}}_{(2,2)\,(j,q)}, \ldots, h^{\mathrm{NL}}_{(p-i,p-i)\,(j,q)}, h^{\mathrm{NL}}_{(p+1-i,p+1-i)\,(j,q)}\right]_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{38}$$

$$\mathbf{M}_{\mathrm{C2C}} = \begin{bmatrix} [\mathbf{M}_{\mathrm{C2CA},1}, \ldots, \mathbf{M}_{\mathrm{C2CA},n_u}] \\ \mathbf{0} \end{bmatrix} \tag{39}$$

$$\mathbf{M}_{\mathrm{C2CA},q} = \begin{bmatrix} \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(1,1)\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(1,1)\,(n_y,q)}\right] \\ \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(1,2)\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(1,2)\,(n_y,q)}\right] \\ \vdots \\ \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(1,p-1)\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(1,p-1)\,(n_y,q)}\right] \\ \vdots \\ \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(M-1,1)\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(M-1,1)\,(n_y,q)}\right] \\ \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(M-1,2)\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(M-1,2)\,(n_y,q)}\right] \\ \vdots \\ \mathrm{diag}\left[\mathbf{H}^{\mathrm{NL}}_{(M-1,p-(M-1))\,(1,q)}, \ldots, \mathbf{H}^{\mathrm{NL}}_{(M-1,p-(M-1))\,(1,q)}\right] \end{bmatrix}^{\mathrm{T}} \tag{40}$$

if $jj < (M-1)$, then $\mathbf{H}^{\mathrm{NL}}$ is calculated as:

$$\mathbf{H}^{\mathrm{NL}}_{(ii,jj)\,(j,q)} = \begin{bmatrix} \mathbf{0}_{((ii+jj-1)\times(M-ii))} & \mathbf{0}_{((ii+jj-1)\times 1)} \\ \mathrm{diag}\left[h^{\mathrm{NL}}_{(1,1+ii)\,(j,q)}, \ldots, h^{\mathrm{NL}}_{(M-ii,M)\,(j,q)}\right] & \mathbf{0}_{((M-ii)\times 1)} \\ \mathbf{0}_{((p+1-M-jj)\times(M-ii))} & \mathbf{0}_{((p+1-M-jj)\times 1)} \end{bmatrix} \tag{41}$$

else if $jj \geq (M-1)$, then $\mathbf{H}^{\mathrm{NL}}$ is calculated as:

$$\mathbf{H}^{\mathrm{NL}}_{(ii,jj)\,(j,q)} = \begin{bmatrix} \mathbf{0}_{((ii+jj-1)\times(M-ii))} & \mathbf{0}_{((ii+jj-1)\times 1)} \\ \mathrm{diag}\left[h^{\mathrm{NL}}_{(1,1+ii)\,(j,q)}, \ldots, h^{\mathrm{NL}}_{(2M-jj-1-ii,2M-jj-1)\,(j,q)}\right] & \mathbf{0}_{((2M-jj-1-ii)\times 1)} \end{bmatrix} \tag{42}$$

$$\mathbf{M}_{\mathrm{C2D}} = \begin{bmatrix} \left[\left[\mathbf{M}_{\mathrm{C2D},1,1}, \ldots, \mathbf{M}_{\mathrm{C2D},M,1}\right], \ldots, \left[\mathbf{M}_{\mathrm{C2D},1,n_u}, \ldots, \mathbf{M}_{\mathrm{C2D},M,n_u}\right]\right] \\ \mathbf{0} \end{bmatrix} \tag{43}$$

$$\mathbf{M}_{\mathrm{C2D},ii,q} = \begin{bmatrix} \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2DVL},ii,1,q}, \ldots, \mathbf{M}_{\mathrm{C2DVL},ii,n_y,q}\right] \\ \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2DVNL},ii,1,q}, \ldots, \mathbf{M}_{\mathrm{C2DVNL},ii,n_y,q}\right] \end{bmatrix}^{\mathrm{T}}_{ii=1,\ldots,M;\, q=1,\ldots,n_u} \tag{44}$$

$$\mathbf{M}_{\mathrm{C2DVL},ii,j,q} = \begin{bmatrix} \mathbf{0}_{((ii-1)\times(p+1-ii))} & \mathbf{0}_{(ii\times(p-ii))} \\ \delta h^{\mathrm{L}}_{(ii,ii)\,(j,q)}\mathbf{I}_{p+1-ii} & h_{\mathrm{ARX}(j)}\delta h^{\mathrm{L}}_{(ii,ii)\,(j,q)}\mathbf{I}_{p-ii} \end{bmatrix}_{ii=1,\ldots,M;\, j=1,\ldots,n_y; q=1,\ldots,n_u} \tag{45}$$

$$\mathbf{M}_{\mathrm{C2DVNL},ii,j,q} = \begin{bmatrix} \mathbf{0}_{((ii-1)\times(p+1-ii))} & \mathbf{0}_{(ii\times(p-ii))} \\ \delta h^{\mathrm{NL}}_{(ii)\,(j,q)}\mathbf{I}_{p+1-ii} & h_{\mathrm{ARX}(j)}\delta h^{\mathrm{NL}}_{(ii)\,(j,q)}\mathbf{I}_{p-ii} \end{bmatrix}_{ii=1,\ldots,M;\, j=1,\ldots,n_y; q=1,\ldots,n_u} \tag{46}$$

$$\mathbf{M}_{\mathrm{C2E}} = \begin{bmatrix} \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2E},1,1}, \ldots, \mathbf{M}_{\mathrm{C2E},n_y,1}\right], \ldots, \mathrm{diag}\left[\mathbf{M}_{\mathrm{C2E},1,n_u}, \ldots, \mathbf{M}_{\mathrm{C2E},n_y,n_u}\right] \\ \mathbf{0} \end{bmatrix} \tag{47}$$

$$\mathbf{M}_{\mathrm{C2E},j,q} = \begin{bmatrix} \mathbf{H}^{\mathrm{VNL}}_{(1,2)\,(j,q)} \\ \mathbf{H}^{\mathrm{VNL}}_{(2,3)\,(j,q)} \\ \vdots \\ \mathbf{H}^{\mathrm{VNL}}_{(M-1,M)\,(j,q)} \\ \mathbf{H}^{\mathrm{VNL}}_{(1,3)\,(j,q)} \\ \mathbf{H}^{\mathrm{VNL}}_{(2,4)\,(j,q)} \\ \vdots \\ \mathbf{H}^{\mathrm{VNL}}_{(M-2,M)\,(j,q)} \\ \vdots \\ \mathbf{H}^{\mathrm{VNL}}_{(1,M)\,(j,q)} \end{bmatrix}^{\mathrm{T}}_{j=1,\ldots,n_y;\, q=1,\ldots,n_u} \tag{48}$$

$$\mathbf{H}^{\mathrm{VNL}}_{(ii,jj)\,(j,q)} = \begin{bmatrix} \mathbf{0}_{((jj-1)\times(p+1-jj))} & \mathbf{0}_{(jj\times(p-jj))} \\ \delta h^{\mathrm{NL}}_{(ii,jj)\,(j,q)}\mathbf{I}_{p+1-jj} & h_{ARX(j)}\delta h^{\mathrm{NL}}_{(ii,jj)\,(j,q)}\mathbf{I}_{p-jj} \end{bmatrix}_{j=1,\ldots,n_y;q=1,\ldots,n_u} \tag{49}$$

The structure of $\mathbf{M}_{\mathrm{C1}}$ is as follows

$$\mathbf{M}_{\mathrm{C1}} = [\mathbf{M}_{\mathrm{C1A}}, \mathbf{M}_{\mathrm{C1B}}, \mathbf{M}_{\mathrm{C1C}}, \mathbf{M}_{\mathrm{C1D}}, \mathbf{M}_{\mathrm{C1E}}] \tag{50}$$

$\mathbf{M}_{\mathrm{C1A}}$ ,$\mathbf{M}_{\mathrm{C1D}}$ and $\mathbf{M}_{\mathrm{C1E}}$ are matrices of zeros of appropriate dimensions

$$\mathbf{M}_{\mathrm{C1CCD}} = [\mathbf{M}_{\mathrm{C1C}}, \mathbf{M}_{\mathrm{C1D}}] \tag{51}$$

In order to obtain the matrix $\mathbf{M}_{\mathrm{C1CCD}}$ a column vector is constructed that contains the Volterra series coefficients according to the following structure:

$$\mathbf{VE} = \begin{bmatrix} \mathbf{VE}_{\mathrm{A1}} \\ \mathbf{VE}_{\mathrm{A2}} \\ \mathbf{VE}_{\mathrm{A3}} \end{bmatrix} \tag{52}$$

$$\mathbf{VE}_{\mathrm{A1}} = \begin{bmatrix} \left[h^{\mathrm{L}}_{(1)\,(1,1)}, \ldots, h^{\mathrm{L}}_{(M)\,(1,1)}\right]^{\mathrm{T}} \\ \vdots \\ \left[h^{\mathrm{L}}_{(1)\,(n_y,1)}, \ldots, h^{\mathrm{L}}_{(M)\,(n_y,1)}\right]^{\mathrm{T}} \\ \left[h^{\mathrm{L}}_{(1)\,(1,2)}, \ldots, h^{\mathrm{L}}_{(M)\,(1,2)}\right]^{\mathrm{T}} \\ \vdots \\ \left[h^{\mathrm{L}}_{(1)\,(n_y,n_u)}, \ldots, h^{\mathrm{L}}_{(M)\,(n_y,n_u)}\right]^{\mathrm{T}} \end{bmatrix} \tag{53}$$

$$\mathbf{VE}_{\mathrm{A2}} = \begin{bmatrix} \left[h^{\mathrm{NL}}_{(1,1)\,(1,1)}, h^{\mathrm{NL}}_{(2,2)\,(1,1)}, \ldots, h^{\mathrm{NL}}_{(M-1,M-1)\,(1,1)}, h^{\mathrm{NL}}_{(M,M)\,(1,1)}\right]^{\mathrm{T}} \\ \vdots \\ \left[h^{\mathrm{NL}}_{(1,1)\,(n_y,1)}, h^{\mathrm{NL}}_{(2,2)\,(n_y,1)}, \ldots, h^{\mathrm{NL}}_{(M-1,M-1)\,(n_y,1)}, h^{\mathrm{NL}}_{(M,M)\,(n_y,1)}\right]^{\mathrm{T}} \\ \left[h^{\mathrm{NL}}_{(1,1)\,(1,2)}, h^{\mathrm{NL}}_{(2,2)\,(1,2)}, \ldots, h^{\mathrm{NL}}_{(M-1,M-1)\,(1,2)}, h^{\mathrm{NL}}_{(M,M)\,(1,2)}\right]^{\mathrm{T}} \\ \vdots \\ \left[h^{\mathrm{NL}}_{(1,1)\,(n_y,n_u)}, h^{\mathrm{NL}}_{(2,2)\,(n_y,n_u)}, \ldots, h^{\mathrm{NL}}_{(M-1,M-1)\,(n_y,n_u)}, h^{\mathrm{NL}}_{(M,M)\,(n_y,n_u)}\right]^{\mathrm{T}} \end{bmatrix} \tag{54}$$

$$\mathbf{VE}_{\mathrm{A3}} = \begin{bmatrix} \mathbf{VE}_{\mathrm{A31},(1,2)} \\ \mathbf{VE}_{\mathrm{A31},(2,3)} \\ \vdots \\ \mathbf{VE}_{\mathrm{A31},(M-1,M)} \\ \mathbf{VE}_{\mathrm{A31},(1,3)} \\ \mathbf{VE}_{\mathrm{A31},(2,4)} \\ \vdots \\ \mathbf{VE}_{\mathrm{A31},(M-2,M)} \\ \mathbf{VE}_{\mathrm{A31},(1,4)} \\ \mathbf{VE}_{\mathrm{A31},(2,5)} \\ \vdots \\ \mathbf{VE}_{\mathrm{A31},(M-3,M)} \\ \vdots \\ \mathbf{VE}_{\mathrm{A31},(1,M)} \end{bmatrix} \tag{55}$$

$$\mathbf{VE}_{\mathrm{A31},(ii,jj)} = \begin{bmatrix} h^{\mathrm{NL}}_{(ii,jj)\,(1,1)} \\ \vdots \\ h^{\mathrm{NL}}_{(ii,jj)\,(n_y,1)} \\ h^{\mathrm{NL}}_{(ii,jj)\,(1,2)} \\ \vdots \\ h^{\mathrm{NL}}_{(ii,jj)\,(n_y,2)} \\ \vdots \\ h^{\mathrm{NL}}_{(ii,jj)\,(1,n_u)} \\ \vdots \\ h^{\mathrm{NL}}_{(ii,jj)\,(n_y,n_u)} \end{bmatrix} \tag{56}$$

After **VE** has been constructed the following code can be used to assign the corresponding values to the vector matrix *index*:

**Algorithm 1**

$k_c = 0$
**for** $ii = 1$ to number of rows of $[\mathbf{VE}]$ **do**
    **for** $ir = 1$ to $n_y \times p$ **do**
        **for** $ic = 1$ to number of columns $[\mathbf{M}_{\mathrm{C2B}}, \mathbf{M}_{\mathrm{C2C}}]$ **do**
            $k_c = k_c + 1$
            **if** $[\mathbf{M}_{\mathrm{C2B}}, \mathbf{M}_{\mathrm{C2C}}]_{ir,ic} = \mathbf{VE}_{ii,1}$ **then**
                $index_{k_c,1} = ic$
            **end if**
        **end for**
    **end for**
**end for**

The elements of the matrix **M**C1CCD are zero except for the following terms:

**Algorithm 2**

**for** $ir = 1$ to $k_c$ **do**

  $\mathbf{M}_{\mathrm{C1CCD}(ir,index(k_c,1))} = k_{ssv}$

**end for**

The effect of the autoregressive terms is included in the interconnection matrix by multiplying the elements in the main diagonal of $\mathbf{M}_{\mathrm{C2AA2}}$ by:

**Algorithm 3**

**for** $i = 1$ to $n_y$ **do**

  **for** $ii = ((i-1) \times p) + 2$ to $(i \times p)$ **do**

    $ir = (nrMA \times (ii-1)) \times ((p \times n_y) + 1 : ncM)$

    $ir = ((p \times n_y) + 1 : ncM) \times (nrMA \times (ii-1))$

    $\mathbf{M}_{\mathrm{C2AAC2}(ii,ii)} = \left(\frac{1}{k_{ssv}}\right) h_{\mathrm{ARX}(i)} \mathbf{M}_{ir} \mathbf{M}_{ic}$

  **end for**

**end for**

where $nrMA$ is equal to the number of rows of the $\mathbf{M}_{\mathrm{A}}$ matrix and $ncM$ is equal to the number of columns of the $\mathbf{M}$ matrix.

After the interconnection matrix has been constructed the terminal condition is accounted for by multiplying all the columns of the $p \times i$, $\forall i \in [1, n_y]$ row of $\mathbf{M}_{C2}$ by $(k_{ssv}/\epsilon_i)$, $\forall i \in [1, n_y]$.

$$\mathbf{M}_{\mathrm{C2}((p\times i),:)} = \frac{k_{ssv}}{\epsilon_i} \mathbf{M}_{\mathrm{C2}((p\times i),:)} \text{ for } i = 1, \ldots, n_y \tag{57}$$

The uncertainty block $\mathbf{\Delta}$ is composed of $nb_{\mathbf{\Delta}}$ different $\mathbf{\Delta}$ sub-blocks where

$$nb_{\mathbf{\Delta}} = 2 + 2Mn_y n_u + n_y n_u \left( \sum_{j=1}^{M-1} \sum_{i=1}^{M-j} (i) \right) \tag{58}$$

the $\mathbf{\Delta}$ sub-blocks are arranged according to the following structure $\mathbf{\Delta} = \mathrm{diag}\left(\mathbf{\Delta}_1, \ldots, \mathbf{\Delta}_{nb_{\mathbf{\Delta}}}\right)$ where the first $nb_{\mathbf{\Delta}} - 1$ blocks are real scalar square matrices related to the uncertainty of the Volterra series coefficients. The dimensions of $\mathbf{\Delta}_1$ are calculated from the following expression:

$$\boldsymbol{\Delta}_1 = (\text{number of columns}\,[\mathbf{M}_{\text{C2C}}]) \times (\text{number of columns}\,[\mathbf{M}_{\text{C2C}}]) \tag{59}$$

The dimensions of the blocks $\boldsymbol{\Delta}_j,\ j \in [2, (Mn_yn_u + 1)]$ are calculated with the following code where $nc\mathbf{M}_{\text{C2DVL},ii,j,q}$ is the number of columns of the matrix block $\mathbf{M}_{\text{C2DVL},ii,j,q}$

**Algorithm 4**

```
du = 1
for j = 1 to n_y do
    for q = 1 to n_u do
        for ii = 1 to M do
            du = du + 1
            Δ_du = ncM_C2DVL,ii,j,q × ncM_C2DVL,ii,j,q
        end for
    end for
end for
```

The dimensions of the blocks $\boldsymbol{\Delta}_j,\ j \in [(Mn_yn_u + 2), (2Mn_yn_u + 1)]$ are calculated with the following code where $nc\mathbf{M}_{\text{C2DVNL},ii,j,q}$ is the number of columns of the matrix block $\mathbf{M}_{\text{C2DVNL},ii,j,q}$

**Algorithm 5**

```
du = M n_y n_u + 1
for j = 1 to n_y do
    for q = 1 to n_u do
        for ii = 1 to M do
            du = du + 1
            Δ_du = ncM_C2DVNL,ii,j,q × ncM_C2DVNL,ii,j,q
        end for
    end for
end for
```

The dimensions of the blocks $\mathbf{\Delta}_j$, $j \in [(2Mn_yn_u + 2), (nb_{\mathbf{\Delta}} - 1)]$ are calculated with the following code where $nc\mathbf{H}^{\text{VNL}}_{(a,b)\,(j,q)}$ is the number of columns of the matrix block $\mathbf{H}^{\text{VNL}}_{(a,b)\,(j,q)}$

**Algorithm 6**

$du = 2Mn_yn_u + 2$
**for** $j = 1$ to $n_y$ **do**
  **for** $q = 1$ to $n_u$ **do**
    **for** $a = 1$ to $(M-1)$ **do**
      **for** $b = 1$ to $(M-a)$ **do**
        $du = du + 1$
        $\Delta_{du} = \left(nc\mathbf{H}^{\text{VNL}}_{(a,b)\,(j,q)}\right) \times \left(nc\mathbf{H}^{\text{VNL}}_{(a,b)\,(j,q)}\right)$
      **end for**
    **end for**
  **end for**
**end for**

The block $\mathbf{\Delta}_j$, $j = nb_{\mathbf{\Delta}}$ is a complex scalar square matrix related to performance of dimensions $(pn_y + 2n_um) \times (pn_y + 2n_um)$.

Figure 1: Control Variable 1 validation sequence for MIMO case

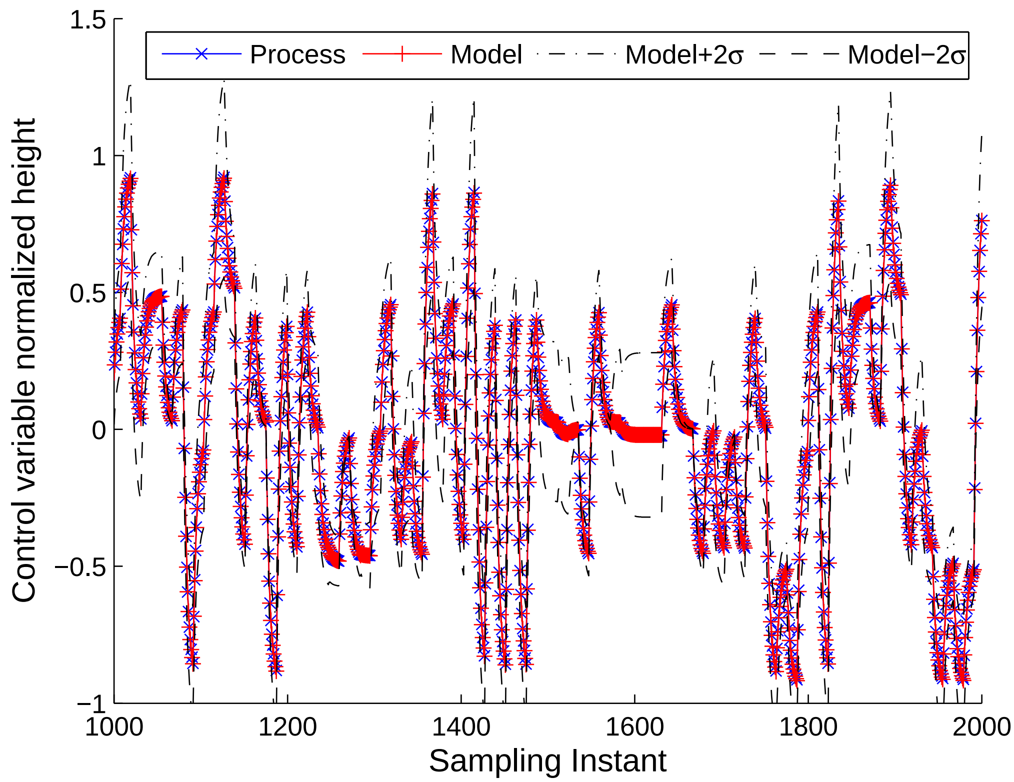

Figure 2: Control Variable 2 validation sequence for MIMO case

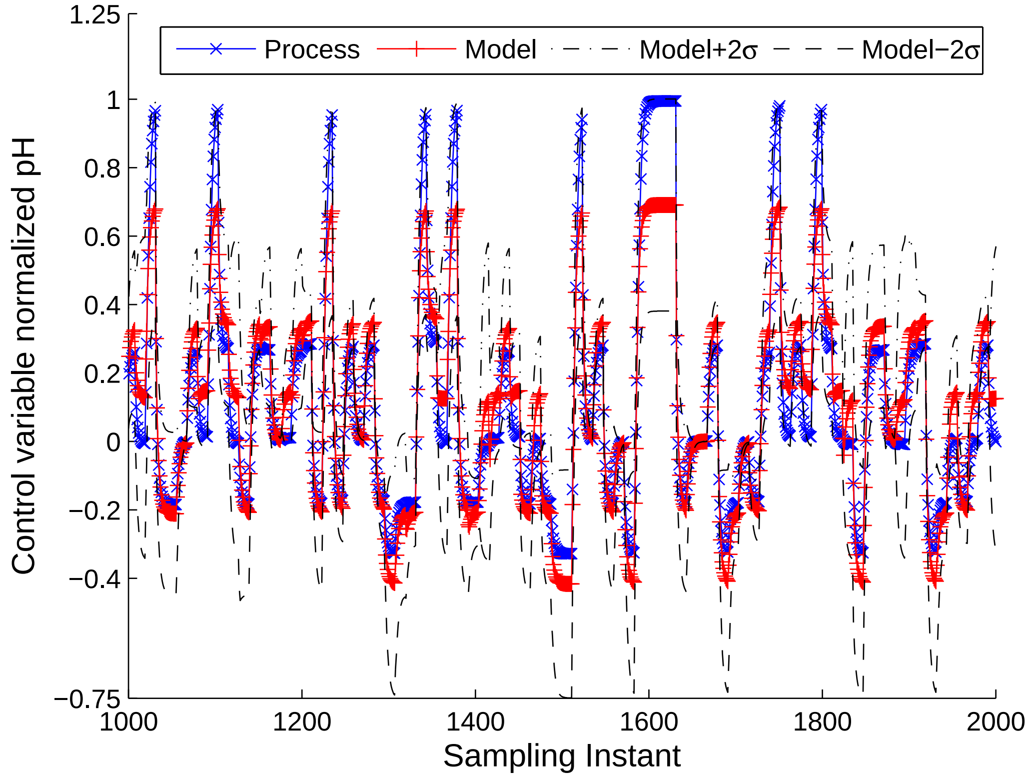

Figure 3: Manipulated variables validation sequence for MIMO case