D'Alembert's Method

- Premise: If $y_1(x)$ is a solution to the ODE, then $y_2 = v(x)\, y_1$ is also a solution.

Example #1

Solve
$$x^2 y'' + 3x y' + y = 0$$
$$y(1) = \frac{1}{2}$$
$$y'(1) = 0$$

CLASSIFICATION
- 2nd order
- linear ODE
- homogeneous
- * Non-constant coeff's
- * No terms missing

By inspection, a solution to the ODE is
$$y_1(x) = \frac{1}{x}$$

Check: $y' = -\frac{1}{x^2}$ $\quad y'' = \frac{2}{x^3}$

And subst into ode
$$x^2 \frac{2}{x^3} - 3x\left(\frac{1}{x^2}\right) + \frac{1}{x} = 0 \quad \text{QED.}$$

Now, using D'Alembert's Method, assume
$$y_2(x) = v(x)\, \frac{1}{x} \leftarrow y_1(x)$$

We need, however, to find $v(x)$ ...

Substitute $y_2(x)$, and its derivatives into the ODE:

$$y_2(x) = \frac{1}{x} v(x)$$

$$\left.\begin{aligned} y_2'(x) &= \frac{1}{x} v'(x) - \frac{1}{x^2} v \\ y_2''(x) &= \frac{1}{x} v''(x) - \frac{2}{x^2} v'(x) + \frac{2v(x)}{x^3} \end{aligned}\right\} \text{Needed for ODE!}$$

The ODE is then:

$$\left(x v'' - 2v' + \cancel{\frac{2v}{x}}\right) + \left(3v' - \cancel{\frac{3v}{x}}\right) + \cancel{\frac{v}{x}} = 0$$

! V term ALLWAYS cancels !

* Leaves 2nd order ODE w. terms missing

$$x v'' + v' = 0 \qquad \text{let } w = v'$$

$$x \frac{dw}{dx} + w = 0 \implies \text{Separate ; Integrate.}$$

$$w = \frac{C_1}{x}$$

Then $\displaystyle \frac{dv}{dx} = w = \frac{C_1}{x} \implies \boxed{v(x) = C_1 \ln x + C_2}$

- Its not over yet, though. We still need to surmise that:

$$y_1(x) = \frac{1}{x}$$

$$y_2(x) = v\,y_1 = C_1 \frac{\ln x}{x} + \frac{C_2}{x}$$

are both solutions to the ODE.

One could apply D'Alembert's method again

$y_3 = v_2(x)\, y_2(x)$

and solve to find $v_2(x) = C_3 + \dfrac{C_4}{C_1 \ln(x) + C_2}$

or, $$y_3(x) = \frac{(C_3C_1)\ln(x) + (C_3C_2) + C_4}{x}$$

or $$y_3(x) = A\,\frac{\ln(x)}{x} + \frac{B}{x}$$

which is the same as $y_2(x)$! Thus we could have stopped there!

Indeed, there is a more rigorous way of checking that we have the complete ~~soln~~ solution; it leads us to Superposition of Solutions and linear independence.